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# On the Computation of Isostables, Isochrons and Other Spectral Objects of the Koopman Operator Using the Dynamic Mode Decomposition

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***Abstract*— Two types of state-space objects - isostables and isochrons - obtained as level sets of Koopman operator eigenfunctions, have recently been shown to be of utility in nonlinear control theory. Algorithms to compute these are in the class of the so-called Dynamic Mode Decomposition (DMD) algorithms or Generalized Laplace Analysis (GLA) algorithms. It is interesting to explore the relationship between these two, which is what we pursue in this paper. We do this in the context more general than isochrons and isostables, deriving results on the relationship of the full Koopman Mode Decomposition with objects computed in DMD, using the fact that GLA is known to be an exact algorithm in the infinite time limit. We also show that finite-dimensional DMD approximations of Koopman eigenfunctions are in the Koopman operator pseudospectrum.**

## I. INTRODUCTION

The Koopman operator framework has proved useful in certain problems in nonlinear control [1], [2]. It has its roots in 1930's through the work of Koopman and von Neumann [3], [4]. The original work by Koopman [3] was devoted to square-integrable observables on the state space of Hamiltonian systems, while the framework we consider here is broader and applies to dissipative systems as well. For a detailed review of this theory and applications, we refer the reader to [**?**].

For a dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

defined on a state-space $M$ (i.e. $\mathbf{x} \in M$ - where we by slight abuse of notation identify a point in a manifold $M$ with its vector representation $\mathbf{x}$ in $\mathbb{R}^m$, $m$ being the dimension of the manifold), where $\mathbf{x}$ is a vector and $\mathbf{F}$ is a possibly nonlinear vector-valued smooth function, of the same dimension as its argument $\mathbf{x}$, denote by $\mathbf{S}^t(\mathbf{x}_0)$ the position at time $t$ of trajectory of (1) that starts at time 0 at point $\mathbf{x}_0$. We call $\mathbf{S}^t(\mathbf{x}_0)$ the flow. Denote by $\mathbf{g}$ an arbitrary, vector-valued observable from $M$ to $\mathbb{R}^k$. The value of this observable $\mathbf{g}$ that the system trajectory starting from $\mathbf{x}_0$ at time 0 sees at time $t$ is

$$\mathbf{g}(t, \mathbf{x}_0) = \mathbf{g}(\mathbf{S}^t(\mathbf{x}_0)). \tag{2}$$

Note that the space of all observables $\mathbf{g}$ is a linear vector space. The family of operators $U^t$, acting on the space of observables parametrized by time $t$ is defined by

$$U^t\mathbf{g}(\mathbf{x}_0) = \mathbf{g}(\mathbf{S}^t(\mathbf{x}_0)). \tag{3}$$

Thus, for a fixed time $\tau$, $U^\tau$ maps the vector-valued observable $\mathbf{g}(\mathbf{x}_0)$ to $\mathbf{g}(\tau, \mathbf{x}_0)$. We will call the family of operators $U^t$ indexed by time $t$ the Koopman operator of the continuous-time system (1). This family was defined for the first time in [3], for Hamiltonian systems. In operator theory, such operators, when defined for general dynamical systems, are often called composition operators, since $U^t$ acts on observables by composing them with the mapping $\mathbf{S}^t$ [5]. In discrete-time the definition is even simpler: if

$$\mathbf{x}' = \mathbf{T}(\mathbf{x}), \tag{4}$$

is a discrete-time dynamical system defined on a set $M$ then the Koopman operator $U$ associated with it is defined by

$$U\mathbf{g}(\mathbf{x}) = \mathbf{g} \circ \mathbf{T}(\mathbf{x}).$$

The operator $U$ is linear, as shown here for the discrete case:

$$\begin{aligned} U(c_1\mathbf{g}_1(\mathbf{x}) + c_2\mathbf{g}_2(\mathbf{x})) &= c_1\mathbf{g}_1(\mathbf{T}(\mathbf{x})) + c_2\mathbf{g}_2(\mathbf{T}(\mathbf{x})) \\ &= c_1U\mathbf{g}_1(\mathbf{x}) + c_2U\mathbf{g}_2(\mathbf{x}). \end{aligned} \tag{5}$$

In the continuous-time case, a similar calculation also shows linearity of members of the Koopman family for each time $t$. Level sets of Koopman operator eigenfunctions play an important role in state-space geometry of nonlinear systems [6]. Two classes of eigenfunctions - isochrons [7], and isostables [8] have been shown to be of importance for nonlinear control problems. In [2] an isostable-based formulation for the problem of convergence to/escape from the equilibrium which is adapted to the short duration of the control, was proposed. In particular, it was shown that the relevant end cost function for the problem to be maximized when the control is switched off is based on the notion of isostables, introduced in [8]. The isostables are sets of the state space that capture the asymptotic behavior of the uncontrolled system. They provide a unique and rigorous measure of how far with respect to time the trajectory is from the equilibrium.

Additionally, isochrons capture the phase of a nonlinear system. An operator-theoretic formulation of isochrons as level sets of a Koopman eigenfunction at purely imaginary eigenvalue is given in [7]. Numerous problems in phase control can be formulated using the notion of isochrons.

In this paper, we examine properties of computation of isostables and isochrons using the so-called Dynamic Mode

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Decomposition (DMD), introduced in [9] and connected to Koopman Mode Decomposition in [10]. Specifically, we show that different versions of DMD yield exact Koopman Modes provided the spectral information (eigenvalues) are exact. To achieve this, we utilize the technique of the Generalized Laplace Analysis [11]. In section II we first show that finite-dimensional approximations of the Koopman operator using companion matrix compute objects in the pseudospectrum of the Koopman operator, and discuss the relationship between the Generalized Laplace Analysis and DMD in both the case when an eigenvalue of the Koopman operator has a negative real part (isostable case) and the case when the real part of an eigenvalue is 0 (isochron case). We conclude in section III.

## II. Spectral Properties of the Koopman Operator

Let $T$ be a dynamical system on a compact metric space $M$. Let $C(M)$ be the space of continuous functions on $M$. Elements of $M$ will be denoted by $x$. The Koopman (or composition) operator $U$ associated with $T$ is defined by

$$Uf(x) = f \circ T(x).$$

For a finite-time evolution of an initial function $f(x) \in C(M)$ under $T$ we get a sequence

$$(f(x), f \circ T(x), ..., f \circ T^{n-1}(x), f \circ T^n(x)).$$

Let $f^i = f \circ T^i(x)$. Then clearly $f^{i+1} = Uf^i$, for $0 \leq i \leq n-1$. If $f^n$ was in the space spanned by $f^0, ..., f^{n-1}$, and these were linearly independent functions, we would have

$$f^n = \sum_{j=0}^{n-1} c_i f^i,$$

for some coefficients $c_i, i = 0, ...n-1$. In that case, the operator $U$ would have a finite-dimensional approximation $\tilde{U}$ on the $\mathrm{span}\{f^0, ...f^{n-1}\}$, given by the companion matrix

$$\tilde{U} = \begin{pmatrix} 0 & 0 & \dots & 0 & c_0 \\ 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{n-1} \end{pmatrix} \tag{6}$$

Let $\tilde{e}$ be an eigenvector of $\tilde{U}$ satisfying

$$\tilde{U}\tilde{e} = \tilde{\lambda}\tilde{e}$$

Then let $\tilde{f} = (f^0, f^1, f^2, ...f^{n-1})^T$. The action of $U$ on $\tilde{e} \cdot \tilde{f}$ is given as

$$\begin{aligned} U\tilde{e} \cdot \tilde{f} &= \tilde{e} \cdot \tilde{f} \circ T = \sum_{i=0}^{n-1} e_i f^i \circ T \\ &= \sum_{i=0}^{n-1} e_i f^{i+1}. \end{aligned} \tag{7}$$

Now, we also have

$$\begin{aligned} \tilde{U}\tilde{e} &= \begin{pmatrix} 0 & 0 & \dots & 0 & c_0 \\ 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{n-1} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} c_0 e_{n-1} \\ e_0 + c_1 e_{n-1} \\ e_1 + c_2 e_{n-1} \\ \vdots \\ e_{n-2} + c_{n-1} e_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ e_0 \\ e_1 \\ \vdots \\ e_{n-2} \end{pmatrix} + e_{n-1} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{pmatrix} = \tilde{\lambda}\tilde{e} \end{aligned} \tag{8}$$

Now using this in (7) we obtain $U\tilde{e} \cdot \tilde{f} = \lambda \tilde{e} \cdot \tilde{f} - e_{n-1}\tilde{c} \cdot \tilde{f} + e_{n-1} f \circ T^n$, or

$$U\tilde{e} \cdot \tilde{f} = \tilde{\lambda}\tilde{e} \cdot \tilde{f} + e_{n-1}(f \circ T^n - \tilde{c} \cdot \tilde{f}). \tag{9}$$

It is now clear that, under the assumption that $f \circ T^n$ is in $\mathrm{span}\{f^0, ...f^{n-1}\}$, $\tilde{e} \cdot \tilde{f}$ is an eigenfunction of $U$. If that assumption is relaxed, and $\tilde{c} \cdot \tilde{f}$ is the orthogonal projection of $f \circ T^n$ to the $\mathrm{span}\{f^0, ...f^{n-1}\}$ as in the companion-matrix version of DMD [10], then $\tilde{e} \cdot \tilde{f}$ is an approximation to the eigenvector of $U$ with an approximate eigenvalue $\tilde{\lambda}$, with the error $e_{n-1}(f \circ T^n - \tilde{c} \cdot \tilde{f}) = e_{n-1} r$, where $r$ is the residual.

Note that the equation (9) could be written as

$$|U\tilde{e} \cdot \tilde{f} - \tilde{\lambda}\tilde{e} \cdot \tilde{f}| = |e_{n-1} r|$$

which means that $\tilde{e} \cdot \tilde{f}$ is in the $(\tilde{\lambda}, \epsilon)$ -pseudospectrum of $U$ for $\epsilon = |e_{n-1} r|$ (see [12]).

### *A. Koopman modes and Generalized Laplace Analysis (GLA)*

One outcome of the Koopman operator theory is the theory of Koopman modes which are the projection of the observables onto Koopman eigenfunctions. These modes correspond to components of the physical field that exhibit exponential growth, decay and/or oscillation in time, and play an important role in the analysis of large systems including fluid flows and buildings. ([11], [13].) The following theorem provides us with a theoretical algorithm for computation of Koopman modes of the observable $f$.

*Theorem 1 (Generalized Laplace Analysis):* Let $f(x, z)$ be a field of observables $f(x, z) : M \times A \to \mathbb{R}^m$, where the observables are indexed over set $A$. We will occasionally drop the dependence on $x$ and denote $f(x, z) = f(z)$ and the iterates of $f$ by $f(T^i x, z) = f^i(z)$. Let $\lambda_0, ..., \lambda_k$ be the simple eigenvalues of $U$ such that $|\lambda_0| \geq |\lambda_1| \geq ... \geq |\lambda_k|$ and there are no other points $\lambda$ in the spectrum of $U$ with

$|\lambda| \geq |\lambda_k|$. Then, the Koopman mode associated with $\lambda_k$ is obtained by computing

$$
\begin{aligned}
f_k &= \phi_k(x)s_k(z) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \lambda_k^{-i}\left(f(T^i x,z) - \sum_{j=0}^{k-1}\lambda_j^i \phi_j(x)s_j(z)\right), \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \lambda_k^{-i}\left(f^i(z) - \sum_{j=0}^{k-1}\lambda_j^i f_j\right), \qquad (10)
\end{aligned}
$$

where $f_j = \phi_j(x)s_j(z)$.

*Proof:* See [14]. ■

In other words, $f_j$ is the skew-projection of the field of observables $f(x,z)$ on the Koopman eigenfunction $\phi_j(x)$ associated with the eigenvalue $\lambda_j$. The skew-projection at eigenvalue $\lambda_l$ is formed by taking out the skew-projection of all $\lambda_j$'s with $|\lambda_j| > |\lambda_l|$.

### *B. Approximation of Koopman modes: Dynamic Mode Decomposition (DMD)*

In this section, we briefly review the basics of Dynamic Mode Decomposition (DMD) and show the explicit relationship to theorem 1. Let

$$
f^0, f^1, f^2, \ldots, f^n \qquad (11)
$$

be a time-sequence of observations on the system. In the companion-matrix method [10], we take the Krylov basis $\{f^0, f^1, ..., f^{n-1}\}$ as the basis for the space of observables (we assume $f$'s to be linearly independent) and represent the action of Koopman operator on that basis by the companion matrix $\tilde{U}$ defined in (6). The entries of companion matrix denoted by $c_i,\ 0 \leq i < n$ are given by approximating the last observation $f^n$ as a linear combination of previous observations, i.e.,

$$
f^n \approx c_0 f^0 + c_1 f^1 + \ldots + c_{n-1} f^{n-1}. \qquad (12)
$$

The coordinates of the observable sequence (11) in the Krylov basis would be

$$
\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}, \begin{pmatrix}c_0\\c_1\\c_2\\\vdots\\c_{n-1}\end{pmatrix}. \qquad (13)
$$

The Dynamic Mode Decomposition of the sequence of observations is then defined to be

$$
f^k = \sum_i v_i \lambda_i^k,\ k < n
$$

where $\lambda_i$'s are *dynamic eigenvalues* (i.e. the eigenvalues of $\tilde{U}$) and $v_i$'s are the *dynamic modes*. The above decomposition can be stated in the matrix form,

$$
\tilde{f} = \tilde{v}\, T \qquad (14)
$$

with $\tilde{v} = (v_0, v_1, v_2, ... v_{n-1})$ and $T$ denoting the Vandermonde matrix,

$$
T = \begin{bmatrix} 1 & \lambda_0 & \lambda_0^2 & \ldots & \lambda_0^{n-1} \\ 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n-1} & \lambda_{n-1}^2 & \ldots & \lambda_{n-1}^{n-1} \end{bmatrix},
$$

In the following theorem, we show that Dynamic Mode Decomposition computes the Koopman modes at $n \to \infty$.

*Theorem 2:* Let $\lambda_i,\ i = 0, 1, \ldots, m$ denote the distinct eigenvalues of the Koopman operator restricted to the $\mathrm{span}\{f_0, f_1, \ldots, f_{n-1}\}$. Also assume that $\lambda_i$ converge to $\mu_i$, the true eigenvalues of the Koopman operator. Then the columns of $\tilde{v}$ in (14) converge to Koopman modes associated with $\lambda_i,\ i = 0, 1, \ldots, m$, as $n \to \infty$.

*Proof:* Without loss of generality assume $\|\lambda_0\| \geq \|\lambda_1\| \geq \|\lambda_2\| \geq \ldots \geq \|\lambda_m\|$. Then one could solve the linear system in (14) as follows: Divide the $i$-th column of the Vandermonde matrix by $\lambda_0^{i-1}$. Then replace the first column by the sum of all columns. The first column of the above system then reads

$$
\begin{aligned}
&f^0 + \frac{1}{\lambda_0} f^1 + \frac{1}{\lambda_0^2} f^2 + \ldots + \frac{1}{\lambda_0^n} f^{n-1} \\
&= (v_0, v_1, v_2, \ldots, v_m) \begin{bmatrix} 1+1+1+\ldots+1 \\ 1 + \frac{\lambda_1}{\lambda_0} + \frac{\lambda_1^2}{\lambda_0^2} + \ldots + \frac{\lambda_1^{n-1}}{\lambda_0^{n-1}} \\ 1 + \frac{\lambda_2}{\lambda_0} + \frac{\lambda_2^2}{\lambda_0^2} + \ldots + \frac{\lambda_2^{n-1}}{\lambda_0^{n-1}} \\ \vdots \\ 1 + \frac{\lambda_{n-1}}{\lambda_0} + \frac{\lambda_{n-1}^2}{\lambda_{n-1}^2} + \ldots + \frac{\lambda_{n-1}^{n-1}}{\lambda_0^{n-1}} \end{bmatrix}.
\end{aligned}
$$

By using the geometric series sum,

$$
\begin{aligned}
&f^0 + \frac{1}{\lambda_0} f^1 + \frac{1}{\lambda_0^2} f^2 + \ldots + \frac{1}{\lambda_0^{n-1}} f^{n-1} \\
&= (v_0, v_1, v_2, \ldots, v_{n-1}) \begin{bmatrix} n \\ \frac{1-(\lambda_1/\lambda_0)^{n-1}}{1-\lambda_1/\lambda_0} \\ \frac{1-(\lambda_2/\lambda_0)^{n-1}}{1-\lambda_2/\lambda_0} \\ \vdots \\ \frac{1-(\lambda_{n-1}/\lambda_0)^{n-1}}{1-\lambda_{n-1}/\lambda_0} \end{bmatrix}.
\end{aligned}
$$

Assume $\lim_{n\to\infty} ||v_n|| = 0$. In the limit $n \to \infty$, we can divide both sides by $n$ to get

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\lambda_0^{-i} f^i &= \lim_{n\to\infty} v_0 + \frac{1}{n}\frac{1-(\lambda_1/\lambda_0)^{n-1}}{1-\lambda_1/\lambda_0} v_1 \\
&+ \frac{1}{n}\frac{1-(\lambda_2/\lambda_0)^{n-1}}{1-\lambda_2/\lambda_0} v_2 + \ldots, \\
&= v_0.
\end{aligned}
$$

where we have used the fact $\|\lambda_0\| \geq \|\lambda_1\| \geq \|\lambda_2\| \geq \ldots$, and used the Koopman-von Neumann Lemma [15] that implies that

$$
\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a(n) = 0 \qquad (15)
$$

provided $\lim_{n\to\infty} a(n) = 0$. Now that we have computed $v_0$, we can treat it as a known variable and move it to the left-hand-side of equation (14), and thus obtain

$$(f^0 - v_0, f^1 - \lambda_0 v_0, f^2 - \lambda_0^2 v_0, \ldots, f^{n-1} - \lambda_0^{n-1} v_0)$$

$$= (v_1, v_2, \ldots, v_m) \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{n-1} & \lambda_{n-1}^2 & \cdots & \lambda_{n-1}^{n-1} \end{bmatrix}.$$

By repeating the same procedure as above, but this time by dividing $\lambda_1^{i-1}$, the second dynamic mode, $v_1$, is computed to be

$$v_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \lambda_1^{-i} (f^i - \lambda_0^i v_0).$$

Using induction, it follows that the $k$-th dynamic mode is given by

$$v_k = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \lambda_k^{-i} (f^i - \sum_{j=0}^{k-1} \lambda_j^i v_j). \tag{16}$$

Comparing equations (10) and (16) indicates the equivalence of Koopman modes and dynamic modes over infinite iterations of the dynamical system. ■

## III. CONCLUSIONS

In this paper, we have studied some problems in computation of spectral objects related to Koopman operators. Two such objects of importance in nonlinear control are isochrons and isostables. We showed that, under the assumption of known eigenvalues for the discrete spectrum associated with the Koopman Mode Decomposition, the computational methods of Dynamic Mode Decomposition compute the Koopman Modes exactly. We also showed that finite-dimensional DMD approximations are in the pseudospectrum of the Koopman operator. The problem of determining under which conditions do the finite-dimensional pseudospectral approximations converge to the true spectrum is tackled in [16].

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