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Controllability of Twist Maps

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Abstract

In this paper we study controllability of a class of discrete-time systems arising in Hamiltonian dynamics. In particular, we consider a two-dimensional integrable twist map with time-dependent (control) perturbation. In contrast to the time independent-perturbation case of the Kolmogorov-Arnold-Moser theorem, no invariant structure survives if the perturbation is made a function of time. Using results from ergodic theory, we show that global controllability is obtained using arbitrary small control input.

1 Introduction

Control of Hamiltonian systems is a topic that has received a lot of attention lately [11]. Besides the intrinsic beauty of the subject, this is due to a number of exciting applications such as satellite control [5], quantum control [6, 7], and control of mixing [8, 12].

In this paper we combine the control-theoretical and dynamical systems point of view to study a class of systems that are well understood from the dynamical systems perspective: perturbations of integrable planar twist maps [16]. These two-dimensional maps defined on an annulus arise from discretization of continuous-time Hamiltonian systems. Integrable twist maps on an annulus have very simple dynamics given by $(x, y) \rightarrow (x + G(y), y)$, $G'(y) > 0$, where x and y are the usual Cartesian coordinates on the plane and x is considered *mod* 1.

Thus, all the initial conditions stay at the same y for all time and $y = const.$ is an invariant manifold for the dynamics. One of the most famous results in dynamical systems, the Kolmogorov-Arnold-Moser (KAM) theorem [1] (in Moser's version [16]) considers a time-independent perturbation of an integrable twist map. Under the condition that every curve $y = const.$ intersects its image under the perturbation, KAM theorem states that the majority of initial conditions stay on 1-dimensional invariant curves close to the unperturbed invariant curves on which $G(y)$ satisfies the Diophantine condition (strong irrationality). It is commonly stated that unperturbed invariant curves that have sufficiently irrational dynamics "persist" under perturbation. The question that we ask here is, how does this change if we allow (bounded) time-dependence of the perturbation? We prove under weak conditions that, for arbitrarily small time-dependent perturbations, every unperturbed invariant curve disappears and global controllability is achieved. This is in marked contrast with the KAM result.

This paper is organized as follows. In Section 2 we discuss the dynamics of twist maps. Section 3 consists of basic definitions. Main results follow in Section 4, simulation results in Section 5, and the conclusions is in Section 6.

2 Dynamics of twist maps

In this section we study the two dimensional integrable twist map F defined on an annulus $\mathcal{A} := \mathbf{S^1} \times [\alpha, \beta]$ and its perturbation. Here $\mathbf{S^1} := R/Z := [0, 1)$ denotes the circle. $\mathbf{R}/Z$ is an equivalence class with $x \equiv y$ if $x - y \in \mathbf{Z}$. A twist map $F : \mathcal{A} \rightarrow \mathcal{A}$ is called integrable if it is of the form

$$F(x, z) = (x + G(z), z), \qquad (1)$$

where $x \in \mathbf{S^1} = \mathbf{R}/\mathbf{Z}$, $z \in [\alpha, \beta]$ with $0 < \alpha \leq y \leq \beta$, and $G'(z) > 0$. Given any initial state or initial condition $(x, z) \in \mathbf{S^1} \times [\alpha, \beta]$, the map F tells us where this initial state will be in the next iterate which is $F(x, z)$. Similarly $F(F(x, z)) = F^2(x, z)$ tells us where $F(x, z)$ will be in the next iterate, and so on. Under the action of this map, each point $x \in \mathbf{S^1}$ is rotated by a monotone function G and point $z \in [\alpha, \beta]$ is mapped onto itself. From the dynamics of the map, it is clear that, for any given $\bar{z} \in [\alpha, \beta]$, the circle of the form $S^1 \times \{z\}$ is invariant under the action of the map; i.e., for any $(x, \bar{z}) \in \mathbf{S^1} \times \{\bar{z}\}$, $F(x, \bar{z}) \in \mathbf{S^1} \times \{\bar{z}\}$.

There are two different types of dynamics are possible on each of these invariant circles. For each rational value of $G(\bar{z}) = \frac{p}{q}$, where p and q are integers, the invariant circle $\mathbf{S^1} \times \{\bar{z}\}$ consists of q periodic orbits. As a matter of fact, if we denote by F^q, the composition of F q times, then it is easy to see that $F^q(x,\bar{z}) = (x+q\frac{p}{q},\bar{z}) = (x,\bar{z})$. However, for irrational values of $G(\bar{z})$, iterates of any initial condition $(x,\bar{z}) \in \mathcal{A}$ are never mapped again to the same point; therefore for all irrational values of $G(\bar{z})$ these invariant circles consist of dense orbits.

Since $G(z)$ is a monotone function, map F can be simplified by defining a new coordinate $y = G(z)$. After the coordinate transformation we get a new map $\hat{F} : \mathcal{A} \to \mathcal{A}$ defined as

$$\hat{F}(x,y) = (x+y,y), \tag{2}$$

where $a = G(\alpha) \leq y \leq G(\beta) = b$. We are interested in studying the dynamics of this map subjected to arbitrary small time-dependent perturbations. Since we want to study the controllability property of the perturbed twist map T, we will write it $T : \mathcal{A} \times U \to \mathcal{A}$ as a discrete time system in the following form:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} x_t + y_t + \epsilon u(t) f(x_t,y_t) \pmod 1 \\ y_t + \epsilon u(t) g(x_t,y_t) \end{pmatrix}, \tag{3}$$

where $x \in \mathbf{S^1}$, $y \in [a,b]$ with $0 < a \leq y \leq b$, and $u \in U = [-1,1]$. Instead of making f and g explicit functions of time, we introduce the time dependence in the form of $u(t)$. We assume that f and g are at least $\mathbf{C^1}$ (differentiable function with continuous derivative) and periodic in x i.e., $f(x+1,y) = f(x,y)$ and $g(x+1,y) = g(x,y)$ and g satisfies following regularity condition. There exists $\delta > 0$ and $\vartheta > 0$ such that

$$\mu\{x : |h(x,y)| > \vartheta\} > \delta \text{ for any fixed } y \in [a,b] \tag{4}$$

This map is thoroughly studied in reference [16] for the case of time-independent perturbations where f and g are analytic and periodic in x. It has been proved that for time-independent perturbations some invariant curves of the form $y = \phi(x) = \phi(x+1)$ will survive, if the perturbed map satisfies the intersection property. The map is said to satisfy the intersection property if a curve y =constant intersects its image curve under the action of the map. We will show that if the perturbation is made a function of time, not only no invariant curve survives, but global controllability can be obtained. In other words, a sequence of control inputs $\{u_t\}$ exists which can steer the system from any given initial state to any final state. As in [16], we also assume that the map satisfies the intersection property, but the global controllability results hold true even if the map does not satisfy it. The intersection property on the map is imposed by assuming that $g(\psi(y),y) = 0$ for some smooth function $\psi : (a,b) \to \mathbf{R}$.

3 Basic definitions

For a discrete time map, the time t takes integer values , $t \in \mathbf{Z}$. Since f and g are assumed to be $\mathbf{C^1}$, the map $T : \mathcal{A} \times U \to \mathcal{A}$ is of class $\mathbf{C^1}$. To be precise, we mean that there exists a $\mathbf{C^1}$ extension of T to an open neighborhood of $\mathcal{A} \times U$. For each fixed $\bar{u}$, the map $T_{\bar{u}} : \mathcal{A} \to \mathcal{A}$ is defined as

$$T_{\bar{u}}(x,y) = T(x,y,\bar{u}).$$

In particular, if we consider the map T_u associated with the map T in (3), we can prove that, for ϵ sufficiently small such that $\epsilon \cdot \max(|\frac{\partial f}{\partial x}|, |\frac{\partial f}{\partial y}|, |\frac{\partial g}{\partial x}|, |\frac{\partial g}{\partial y}|) < 1$, the map T is invertible for each fixed u. This can be verified as follows. For each fixed u, we have $T_u(x_t,y_t) = (x_{t+1},y_{t+1})$. Let $\hat{T}_u(x,y,x',y') := T_u(x,y) - (x',y')$ then $\hat{T}_u(x_t,y_t,x_{t+1},y_{t+1}) = 0$. If $\det|\frac{\partial \hat{T}_u}{\partial(x,y)}| \neq 0$, then by implicit function theorem we know that there exists an open neighborhood $\mathcal{N}$ of $(x',y') = (x_{t+1},y_{t+1})$ and a unique function $T_u^- : \mathcal{A} \to \mathcal{A}$ such that $\hat{T}_u(T_u^-(x',y'),x',y') = 0$. So, we need to verify that $\det|\frac{\partial \hat{T}_u}{\partial(x,y)}| \neq 0$

$$\begin{aligned}\det|\frac{\partial \hat{T}_u}{\partial(x,y)}| &= \begin{vmatrix} 1+\epsilon u(t)\frac{\partial f}{\partial x} & 1+\epsilon u(t)\frac{\partial f}{\partial y} \\ \epsilon u(t)\frac{\partial g}{\partial x} & 1+\epsilon u(t)\frac{\partial g}{\partial y} \end{vmatrix} \\ &= 1+\epsilon u(t)(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}) + O(\epsilon^2) \neq 0.\end{aligned}$$

So for sufficiently small ϵ map T is invertible. Since for each fixed u, T_u is invertible we can associate an inverse time map with equation

$$x_t = T_u^-(x_{t+1}).$$

The maps T_u and T_u^- can be considered as one-step -forward and one-step-backward maps, respectively. The composition of these maps obtained by applying sequence of control inputs $u_0,..u_k$ is denoted respectively by

$$T_{u_k,..,u_0} = T_{u_k} \circ .. \circ T_{u_0}$$

and

$$T^-_{u_k,..,u_0} = T^-_{u_0} \circ .. \circ T^-_{u_k}.$$

Let $S_k^+(x)$ be the set of points attainable from x in k forward steps and $S^+(x)$ the set of points attainable from x in any positive number of forward steps. Let $S_k^-(x)$ be the set of points controllable to x in k forward steps and $S^-(x)$ be the set of points controllable to x in any positive number of steps.

Definition 3.1 *The system is said to be controllable if for any given initial state (x_0,y_0) and any final state (x_f,y_f) there exists a sequence of control inputs $u_0,..,u_k$ such that $T_{u_k,..,u_0}(x_0,y_0) = (x_f,y_f)$.*

Definition 3.2 *The system is backward accessible from x if the set of points controllable to x (i.e., $S^-(x)$) has a nonempty interior. The system is said to be backward accessible if it is backward accessible from all points.*

4 Main result

Theorem 4.1 *The invertible twist map (3) with f and g $\mathbf{C}^1$ functions satisfying the intersection property and g satisfying the regularity condition (4) is globally controllable for arbitrary small ϵ if and only if every invariant manifold of the unperturbed map ($\epsilon = 0$) is not invariant for the perturbed map ($\epsilon \neq 0$).*

Proof : We will prove the necessary part first. Assume that the condition in the theorem is not satisfied i.e., there exists an invariant manifold $\mathcal{M}$ of the unperturbed map which remains invariant for the perturbed map. Consider any initial state $(x_0, y_0) \in \mathcal{M}$, then $T_{u_k,..,u_0}(x_0, y_0) \in \mathcal{M}$ for any sequence of control inputs $\{u_t\}$ by definition of invariant manifold. Hence, any initial state $(x, y) \in \mathcal{M}$ is uncontrollable.

To prove the sufficient condition, we will use the following proposition.

Proposition 4.2 *The invertible twist map (3) satisfying the intersection property is globally controllable for arbitrary small ϵ if and only if the following controllability conditions are satisfied.*

1. *For any given fixed $y \in [a, b]$, g does not vanish identically.*
2. *For all $y_{Q\setminus Z} \in \mathbf{Q} \setminus \mathbf{Z}$:*

   If $g(\bar{x}, y_{Q\setminus Z}) = 0$, then let $A_k(\bar{x}) = \{x_k : x_k = \bar{x} + k y_{Q\setminus Z} + \epsilon u(k-1) f(x_{k-1}, y_{Q\setminus Z})\}$. $A_k(\bar{x})$ is the set of all points which can be reached from $\bar{x}$ on k^{th} iterate with fixed $y = y_{Q\setminus Z}$. We assume that there exists an integer $k_1' \in \mathbf{Z}^+$ such that $x' \in A_{k_1'}(\bar{x})$ and $g(x', y_{Q\setminus Z}) \neq 0$ for the first time.
3. *For all $y_Z \in \mathbf{Z}$:*

   (a) *f and g does not vanish simultaneously; i.e., if $f(x_1, y_Z) = 0$ and $g(x_2, y_Z) = 0$, then $x_1 \neq x_2$.*

   AND

   (b) *If $g(\bar{x}, y_Z) = 0$, then let $B_k(\bar{x}) = \{x_k : x_k = \bar{x} + \epsilon u(k-1) f(x_{k-1}, y_Z)\}$, we assume that there exists an integer $k_2 \in \mathbf{Z}^+$ such that $x' \in B_{k_2'}(\bar{x})$ and $g(x', y_Z) \neq 0$ for the first time.*

Proof :We will prove the sufficient part first. We will show that if the condition in the theorem is satisfied, then the system is controllable. To prove this, we will make use of two lemmas:

Lemma 4.3 *The invertible twist map (3) is backward accessible*

Proof : Consider any point $(x_f, y_f) \in \mathcal{A}$. We have to show that the set of all point controllable to (x_f, y_f) contains an open set. Let

$$\mathcal{U} = T^-_{u_1,u_0}(x_f, y_f) = T^-_{u_1} \circ T^-_{u_0}(x_f, y_f), \quad (5)$$

so

$$\begin{aligned}\mathcal{U} = \{&(x, y) : x + 2y + \epsilon u_0(f(x, y) + g(x, y)) + \\ &\epsilon u_1 f(x_1, y_1) - x_f = 0,\ y + \epsilon u_0 g(x, y) + \epsilon u_1 g(x_1, y_1) \\ &-y_f = 0, \text{with } u_0 \in [-1, 1] \text{ and } u_1 \in [-1, 1]\}, \end{aligned} \quad (6)$$

where

$$x_1 = x + y + \epsilon u_0 f(x, y), \quad y_1 = y + \epsilon u_0 g(x, y).$$

Let

$$f_1(x, y, u_0, u_1) = x + 2y + \epsilon u_0(f(x, y) + g(x, y)) + \epsilon u_1 f(x_1, y_1) - x_f \quad (7)$$

$$f_2(x, y, u_0, u_1) = y + \epsilon u_0 g(x, y) + \epsilon u_1 g(x_1, y_1) - y_f, \quad (8)$$

then $f_1(x_f - 2y_f, y_f, 0, 0) = 0$ and $f_2(x_f - 2y_f, y_f, 0, 0) = 0$ and

$$\frac{D(f_1, f_2)}{D(x, y)} = \left| \begin{array}{cc} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{array} \right|_{(x_f - 2y_f, y_f, 0, 0)} = 1 + O(\epsilon) \neq 0$$

Hence, by the implicit function theorem, there exists an open neighborhood $\mathcal{O}$ of $(u_0, u_1) = (0, 0)$ and unique functions Ψ_1 and Ψ_2 defined on $\mathcal{O}$ and taking values in $\mathbf{R}$ such that

$$f_1(\Psi_1(u_0, u_1), \Psi_2(u_0, u_1), u_0, u_1) = 0 \quad (9)$$

$$f_2(\Psi_1(u_0, u_1), \Psi_2(u_0, u_1), u_0, u_1) = 0 \quad (10)$$

for all $(u_0, u_1) \in \mathcal{O}$. This proves that the inverse image of (x_f, y_f) contains an open set. ■

Lemma 4.4 *Given any initial state $(x_0, y_0) \in \mathcal{A}$, there exists a finite sequence of control inputs $\{u_0, .., u_{\ell-1}\}$ such that y_ℓ is irrational and arbitrarily close to y_0, where $(x_\ell, y_\ell) = T_{u_{\ell-1},...,u_0}(x_0, y_0)$.*

Proof: If y_0 is irrational, then $\ell = 0$. If y_0 is rational, then we can consider two different cases: $g(x_0, y_0) = 0$ and $g(x_0, y_0) \neq 0$.

- When $g(x_0, y_0) \neq 0$, then $y_1 = y_0 + \epsilon u(0) g(x_0, y_0)$ can be chosen to be irrational by properly selecting the value of $u(0)$. Since irrational numbers are dense in $[0,1)$, y_1 can be made arbitrarily close to y_0 by making $u(0)$ sufficiently small. So for this case $\ell = 1$.
- When y_0 is rational and $g(x_0, y_0) = 0$, we can again consider two different cases : $y_0 \in \mathbf{Q} \setminus \mathbf{Z}$ and $y_0 \in \mathbf{Z}$.
  - When $y_0 \in \mathbf{Q} \setminus \mathbf{Z}$ and $g(x_0, y_0) = 0$, then we know that there exists an integer $k_1' \in \mathbf{Z}^+$ such that $x' \in A_{k_1'}(x_0)$ and $g(x', y_{Q\setminus Z}) \neq 0$ for the first time. Since k_1' is such that $g(x_{k_1'}, y_{Q\setminus Z}) \neq 0$,for the first time $y_{k_1'} = y_0$. With $g(x_{k_1'}, y_{Q\setminus Z}) \neq 0$, $y_{k_1'+1}$ can be made irrational with proper choice of $u(k_1')$ since
  $$y_{k_1'+1} = y_{k_1'} + \epsilon u(k_1') g(x_{k_1'}, y_{Q\setminus Z}).$$
  $y_{k_1'+1}$ can be made arbitrarily close to y_0 by making $u(k_1')$ sufficiently small and hence $\ell = k_1' + 1$ for this case.
  - When $y_0 \in \mathbf{Z}$ and $g(x_0, y_0) = 0$, then we know that f and g do not vanish simultaneously and there exists an integer $k_2' \in \mathbf{Z}^+$ and $x' \in B_{k_2'}(x_0)$ such that $g(x', y_Z) \neq 0$. With $g(x_{k_2'}, y_0) \neq 0$ and $y_{k_2'}$ can be made irrational with proper choice of $u(k_2')$
  $$y_{k_2'+1} = y_{k_2'} + \epsilon u(k_2') g(x_{k_2'}, y_0)$$
  where $y_{k_2'+1}$ can be made arbitrarily close to $y_0 = y_{k_2'}$ by making $u(k_2')$ sufficiently small. So $\ell = k_2' + 1$.

■

The control strategy consists of turning on the input (either positive or negative depending upon whether $y_0 < y_f$ or $y_0 > y_f$ and $g(x,y) > 0$ or $g(x,y) < 0$) whenever $|g(x,y)| > \vartheta$ until y_k belongs to small neighborhood of y_f . Once y_k is steered to the neighborhood of y_f input will be made zero till x_k is steered to neighborhood of x_f.

Proof of the Proposition 4.2 : Let (x_0, y_0) and (x_f, y_f) be the initial and final state, respectively. Since the system is backward accessible by Lemma 4.3, the set of points $\mathcal{U}$ controllable to (x_f, y_f) contains an open set; hence there exists an open rectangle $\mathcal{V} \subset \mathcal{U}$. Let $\pi_1(\mathcal{V}) = \mathcal{V}_x$ and $\pi_2(\mathcal{V}) = \mathcal{V}_y$ where π is the projection map and $\pi_i(x_1, x_2) = x_i$ for $i = 1, 2$. We will show that there exists a sequence of inputs $\{u_t\}$ such that $\pi_1(T_{u_k,..,u_1}(x_0, y_0)) \in \mathcal{V}_x$ and $\pi_2(T_{u_k,..,u_1}(x_0, y_0)) \in \mathcal{V}_y$. Starting with initial state (x_0, y_0), we know by Lemma 4.4 that there exists an integer ℓ such that y_ℓ is irrational. Let $\bar{y} \in \mathcal{V}_y$ be such that $\bar{y} - y_\ell =: p$ is a rational number and let $m \in \mathbf{Z}^+$ be such that $\frac{p}{m} = \alpha \in (-\epsilon\vartheta, \epsilon\vartheta)$. With y_ℓ irrational the orbit of the rotation map given by

$$x_{t+1} = x_t + y_t \pmod 1$$

is dense in $[0,1)$ for $y_t = y_\ell$; hence, there exists an integer $k_1 - 1$ such that $|g(x_{\ell+k_1-1}, y_{\ell+k_1-1})| > \vartheta$ since the set of points $\{x : |g(x,y)| > \vartheta\}$ is of positive measure by Lemma 4.4. With $|g(x_{\ell+k_1-1}, y_{\ell+k_1-1})| > \vartheta$ input $u(\ell + k_1 - 1)$ can be chosen so that

$$y_{\ell+k_1} = y_{\ell+k_1-1} + \alpha = y_\ell + \alpha.$$

Since control input $u(t) = 0$ for $t = \ell, .., \ell + k_1 - 1$, it follows that $y_{\ell+k_1-1} = y_\ell$. Now $y_{\ell+k_1}$ is still irrational because $y_{\ell+k_1-1}$ is irrational and α is rational. So again the orbit of the rotation map given by

$$x_{t+1} = x_t + y_t \pmod 1$$

is dense in $[0,1)$; hence, there exists an integer k_2 such that $|g(x_{\ell+k_1+k_2-1}, y_{\ell+k_1+k_2-1})| > \vartheta$ and, with proper choice of $u(\ell + k_1 + k_2 - 1)$, we have

$$y_{\ell+k_1+k_2} = y_{\ell+k_1+k_2-1} + 2\alpha = y_{\ell+k_1} + 2\alpha.$$

$y_{\ell+k_1+k_2}$ is still irrational; hence the above procedure can be repeated $m - 2$ times to get

$$y_K = \bar{y}$$

where $K = \ell + \sum_{i=1}^{m} k_i$. Now $y_K = \bar{y} \in \mathcal{V}_y$ is also irrational. With y_K irrational, the orbit of the rotation map given by

$$x_{t+1} = x_t + y_t \pmod 1$$

is dense in $[0,1)$ and hence there exists an integer n such that $x_{K+n} \in \mathcal{V}_x$. With $x_{K+n} \in \mathcal{V}_x$ and $y_{K+n} = y_K \in \mathcal{V}_y$, $(x_{K+n}, y_{K+n}) \in \mathcal{V} \subset \mathcal{U}$. Since all the points of $\mathcal{U}$ are controllable to (x_f, y_f), the system is controllable.

Now we will show that the conditions in the proposition are also necessary for the controllability. Assume that the Condition 1 is not true; i.e., there exists a $\tilde{y} \in [a, b]$ such that $g(x, \tilde{y}) = 0$ for all $x \in \mathbf{S}^1$. Consider any initial state which is of the form $(x_0, y_0) = (x_0, \tilde{y})$. Then

$T_{u_k,..,u_0}(x_0, \tilde{y}) = (x_{k+1}, \tilde{y})$ for any sequence of control inputs since

$$\begin{aligned} x_{k+1} &= x_k + \tilde{y} + \epsilon u(k) f(x_k, \tilde{y}) \\ y_{k+1} &= \tilde{y}. \end{aligned} \quad (11)$$

Hence, any initial condition $(x_0, \tilde{y})$ with $x_0 \in \mathbf{S}^1$ is uncontrollable.

Now assume that the condition 2 is not satisfied; i.e., there exists an $\tilde{y} \in \mathbf{Q} \setminus Z$ and $\tilde{x} \in \mathbf{S}^1$ for which there exists no $k \in \mathbf{Z}^+$ and $x' \in A_k(\tilde{x})$ such that $g(x', \tilde{y}) \neq 0$. Then the initial state of the form $(x_0, y_0) = (\tilde{x}, \tilde{y})$ is uncontrollable because $y_k = \tilde{y}$ for all k.

Assume that the condition 3(a) is not true; i.e., there exists $\tilde{x} \in \mathbf{S}^1$ and $\tilde{y} \in \mathbf{Z}$ such that $f(\tilde{x}, \tilde{y}) = g(\tilde{x}, \tilde{y}) = 0$; then

$$\begin{aligned} x_1 &= \tilde{x} + \tilde{y} + \epsilon u(0) f(\tilde{x}, \tilde{y}) (\text{mod } 1) = \tilde{x} \\ y_1 &= \tilde{y} + \epsilon u(0) g(\tilde{x}, \tilde{y}) = \tilde{y}. \end{aligned} \quad (12)$$

So $(\tilde{x}, \tilde{y})$ is a fixed point of the map and the system is uncontrollable from $(\tilde{x}, \tilde{y})$.

Assume that the condition 3(a) is true but condition 3(b) is not true; i.e., f and g do not vanish simultaneously but there exists an $\tilde{x} \in \mathbf{S}^1$ for which there exists no $k \in \mathbf{Z}^+$ and $x' \in B_{k'_2(\tilde{x})}$ such that $g(x', \tilde{y}) \neq 0$. So $y_k = \tilde{y}$ for all k and the system is uncontrollable from $(x_0, y_0) = (\tilde{x}, \tilde{y})$. ■

Proof of the main Theorem 4.1 cont. : To prove the sufficient part we need to show that if no invariant manifold for the unperturbed map is invariant for the perturbed map then the system is controllable. By proposition 4.2 we know that system is controllable if and only if controllability conditions are satisfied. So the sufficient part of the main theorem is equivalent to proving that if the controllability conditions are not satisfied then there exists an invariant manifold for the perturbed map.

By proposition 4.2, we know that if the condition 1 is not satisfied then all initial states of the form $(x_0, y_0) = (x, \tilde{y})$ are uncontrollable, where $x \in \mathbf{S}^1$ and $\tilde{y}$ is such that $g(x, \tilde{y}) = 0$ for all $x \in \mathbf{S}^1$. Hence there exists an invariant manifold of the form $\{\tilde{y}\} \times \mathbf{S}^1$.

If Condition 2 in proposition 4.2 is not satisfied, then there exists a $\tilde{y} \in \mathbf{Q} \setminus Z$ and $\tilde{x} \in \mathbf{S}^1$ for which there exist no $k \in \mathbf{Z}^+$ and $x' \in A_k(\tilde{x})$ such that $g(x', \tilde{y}) \neq 0$. Hence, $y_k = \tilde{y}$ for all k and there exists an invariant manifold of the form $\{\tilde{y}\} \times \mathcal{N}_1$, where $\mathcal{N}_1 = \bigcup_{k=1}^{\infty} A_k(\tilde{x})$. When condition 3(a) is not satisfied, then there exists a fixed point which is an invariant manifold of dimension zero. When condition 3(b) is not satisfied, then there exists $\tilde{y} \in \mathbf{Z}$ and $\tilde{x} \in \mathbf{S}^1$ for which there exists no $k \in \mathbf{Z}^+$ and $x' \in B_k(\tilde{x})$ such that $g(x', \tilde{y}) \neq 0$. Hence, $y_k = \tilde{y}$ for all k and there exists an invariant manifold of the form $\{\tilde{y}\} \times \mathcal{N}_2$, where $\mathcal{N}_2 = \bigcup_{k=1}^{\infty} A_k(\tilde{x})$. ■

5 Conclusions

With the combination of ergodic theory results and controllability results from control theory, we have proved the controllability of a discrete time nonlinear map using arbitrary small control input. The KAM result hold true for time-independent perturbation. We proved that when the perturbation is made a function of time, under weak conditions complete controllability is obtained.

The control strategy that we pursue stems from [10], where natural dynamics of the system is used to achieve controllability on groups. Given that phase space integrable Hamiltonian systems are foliated by lower dimensional tori, these methods prove quite useful. Generalization to n-degrees of freedom Hamiltonian systems is being currently pursued.

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