Koopman Spectrum for Cascaded Systems

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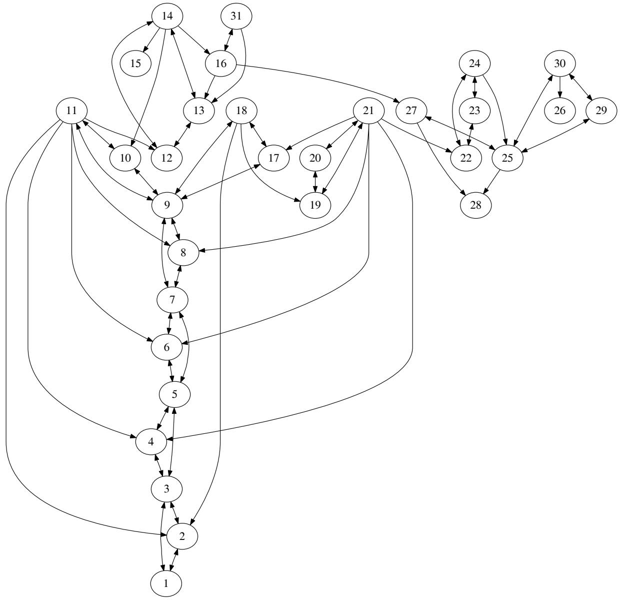
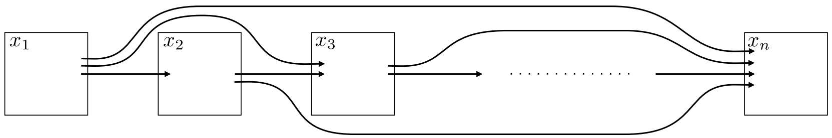
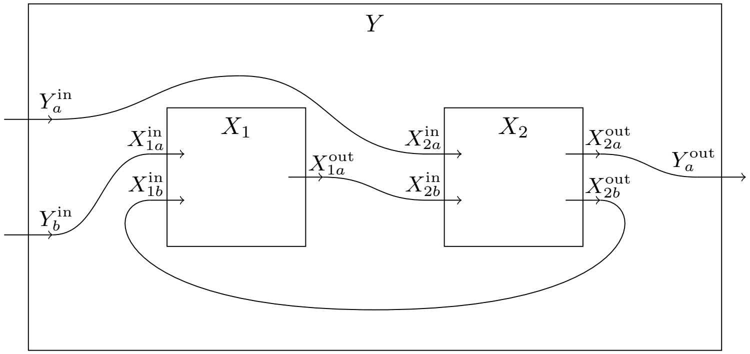
Systems of Systems (SoS)

- **Systems of Systems (SoS)**
  - Submodules wired together (composed) to form more complex systems
- SOS examples come from:
  - Engineered mechanical/electrical systems
  - Networks
    - Chemical-Biological
    - Information
- **Analysis and Prediction** of SoS behavior can be **hard with traditional tools**
  - Geometric methods restricted to low-dimensions
  - Simulation memory requirements can be intractable
- **Tools needed** to analyze behavior of observables on composed systems without simulation





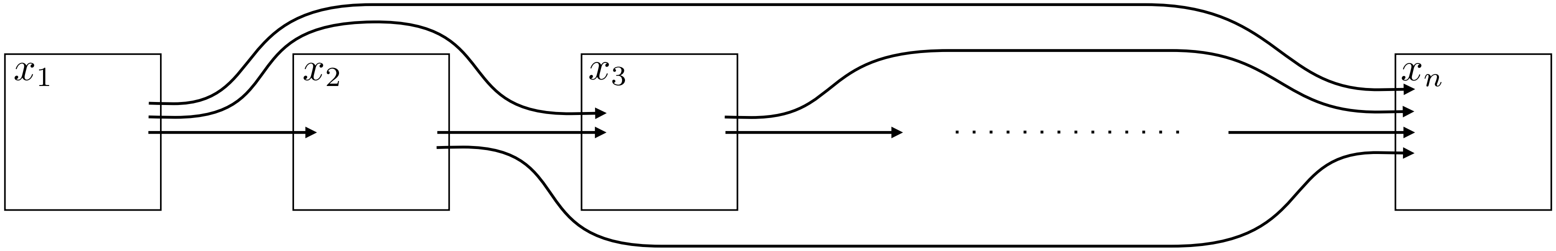


Decomposition of system into cascade structure

- Many systems (engineered and natural) exhibit a structure of a **forward production** unit with slower feedback loops
  - A number of algorithms have been proposed to decompose into interconnected components (& analyze)
    - Callier et al, 1976
    - Pichai, et al, 1983 (Graph theoretic Hierarchical decomposition)
    - Mezic, 2004 (Horizontal-Vertical decomposition)
    - Mesbahi, Haeri, 2015 (Block triangular, Block diagonal form)

- Forward production unit has a **cascade structure**
  - Downstream systems do not affect upstream systems

- **Goal is to understand the behavior of the forward production unit (cascade structure)**



Koopman principal eigenfunctions

- Spectral analysis of the Koopman operator indicates how observables on a system behave
  - Principal eigenvalues generate the entire point spectrum of the operator

$$x(t+1) = Ax(t) + N(x(t))$$

$$Av_i = \lambda_i v_i \qquad \langle v_i, w_j \rangle = \delta_{i,j}$$

- Principal eigenfunctions

$$\psi_j(x) = \langle x, w_j \rangle$$

Principal eigenvalue

$$\lambda_i$$

- Generate new eigenfunctions

$$\phi(x) = \psi_1(x)^{k_1} \cdots \psi_n(x)^{k_n}$$

Product of eigenvalues

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n}$$

How do these fundamental objects change when systems are wired together?

Koopman Spectrum for Cascaded Systems

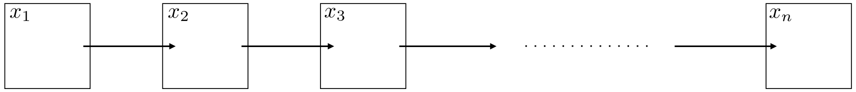
Outline

1. Asymptotic equivalence and zero relative error between linear cascade and nominal system
2. Conservation of principal eigenvalues, modification of principal eigenfunctions
3. Push results for cascades of linear systems with linear connections to cascades of nonlinear systems

(Chained) Linear cascade and nominal systems

(Chained) Linear Cascaded system (Lin)

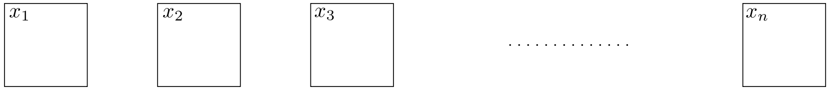
$$x_1(t+1) = L_1 x_1(t)$$
$$x_i(t+1) = L_i x_i(t) + C_{i,i-1} x_{i-1}(t)$$



$$\text{Lin}^{\circ t}(x_1,\ldots,x_n) = (\mathcal{L}_1^{\circ t}(x_1), \mathcal{L}_2^{\circ t}(x_1,x_2), \ldots, \mathcal{L}_n^{\circ t}(x_1,\ldots,x_n))$$

Nominal system (Nom)

$$C_{i,j} = 0$$



$$\text{Nom}^{\circ t}(x_1,\ldots,x_n) = (L_1^t(x_1), L_2^t(x_2), \ldots, L_n^t(x_n))$$

Assumptions

(i) *L_i is invertible and diagonalizable for all $i = 1,\ldots,n$,*

$$L_i V_i = V_i \Lambda_i. \tag{35}$$

(ii) *(Disjoint spectrums) The spectrums of each layer are pairwise disjoint. That is for $i, j \in \{1,\ldots,n\}$ satisfying $i \neq j$*

$$\sigma(L_i) \cap \sigma(L_j) = \emptyset. \tag{36}$$

(iii) $\|L_1\| < \|L_2\| < \cdots < \|L_n\| \le 1.$

Solutions of the cascade system

The orbit in the ith system is

$$x_i(t) = \Pi_i \circ \mathsf{Lin}^{\circ t}(x_1, \ldots, x_n) = L_i^t \mathsf{pert}_i(x_1, \ldots, x_i) + \sum_{j=1}^{i-1} (-1)^{i-j} D_{i,j} L_j^t \mathsf{pert}_j(x_1, \ldots, x_j)$$

Proof: By induction

Solutions of the cascade system

The orbit in the ith system is

$$x_i(t) = \Pi_i \circ \mathsf{Lin}^{\circ t}(\underbrace{x_1, \ldots, x_n}_{\text{Initial condition in the cascaded linear system}}) = L_i^t \mathsf{pert}_i(x_1, \ldots, x_i) + \sum_{j=1}^{i-1} (-1)^{i-j} D_{i,j} L_j^t \mathsf{pert}_j(x_1, \ldots, x_j)$$

Solutions of the cascade system

The orbit in the i^th system is

$$x_i(t) = \Pi_i \circ \mathsf{Lin}^{\circ t}(x_1, \dots, x_n) = L_i^t \underbrace{\mathsf{pert}_i(x_1, \dots, x_i)} + \sum_{j=1}^{i-1} (-1)^{i-j} D_{i,j} L_j^t \underbrace{\mathsf{pert}_j(x_1, \dots, x_j)}$$

Perturbed initial conditions for nominal system

$$\mathsf{pert}_1(x_1) = x_1$$

$$\mathsf{pert}_i(x_1, \dots, x_i) = x_i + \sum_{j=1}^{i-1} (-1)^{i-1-j} D_{i,j} \mathsf{pert}_j(x_1, \dots, x_j)$$

$$\mathsf{pert}(x_1, \dots, x_n) = (\mathsf{pert}_1 \circ \Pi_1, \mathsf{pert}_2 \circ (\Pi_1, \Pi_2), \dots, \mathsf{pert}_{n-1} \circ (\Pi_1, \dots, \Pi_{n-1}), \mathsf{pert}_n)(x_1, \dots, x_n)$$

Lower triangular structure (e.g. 3 layer cascade)

$$\mathsf{pert}(x_1, x_2, x_3) = \begin{bmatrix} \mathsf{pert}_1(x_1) \\ \mathsf{pert}_2(x_1, x_2) \\ \mathsf{pert}_3(x_1, x_2, x_3) \end{bmatrix} = \underbrace{\begin{bmatrix} I_1 & & \\ 0 & I_2 & \\ -D_{3,1} & D_{3,2} & I_3 \end{bmatrix} \underbrace{\begin{bmatrix} I_1 & & \\ D_{2,1} & I_2 & \\ 0 & 0 & I_3 \end{bmatrix} \underbrace{\begin{bmatrix} I_1 & & \\ 0 & I_2 & \\ 0 & 0 & I_3 \end{bmatrix}}_{\mathsf{pert}_1}}_{\mathsf{pert}_2}}_{\mathsf{pert}_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solutions of the cascade system

The orbit in the i^{th} system is

$$x_i(t) = \Pi_i \circ \mathsf{Lin}^{\circ t}(x_1, \ldots, x_n) = L_i^t \mathsf{pert}_i(x_1, \ldots, x_i) + \sum_{j=1}^{i-1} (-1)^{i-j} D_{i,j} \underbrace{L_j^t \mathsf{pert}_j(x_1, \ldots, x_j)}_{\text{Evolution of perturbed i.c. due to nominal system}}$$

Solutions of the cascade system

The orbit in the i^{th} system is

$$x_i(t) = \Pi_i \circ \mathsf{Lin}^{\circ t}(x_1, \dots, x_n) = L_i^t \mathsf{pert}_i(x_1, \dots, x_i) + \sum_{j=1}^{i-1} (-1)^{i-j} \underbrace{D_{i,j}}_{\text{Map } j^{\text{th}} \text{ nominal system orbit into } i^{\text{th}} \text{ system}} L_j^t \mathsf{pert}_j(x_1, \dots, x_j)$$

$$D_{i,i} = I_{d_i} \qquad \forall i \in \{1, \dots, n\}$$

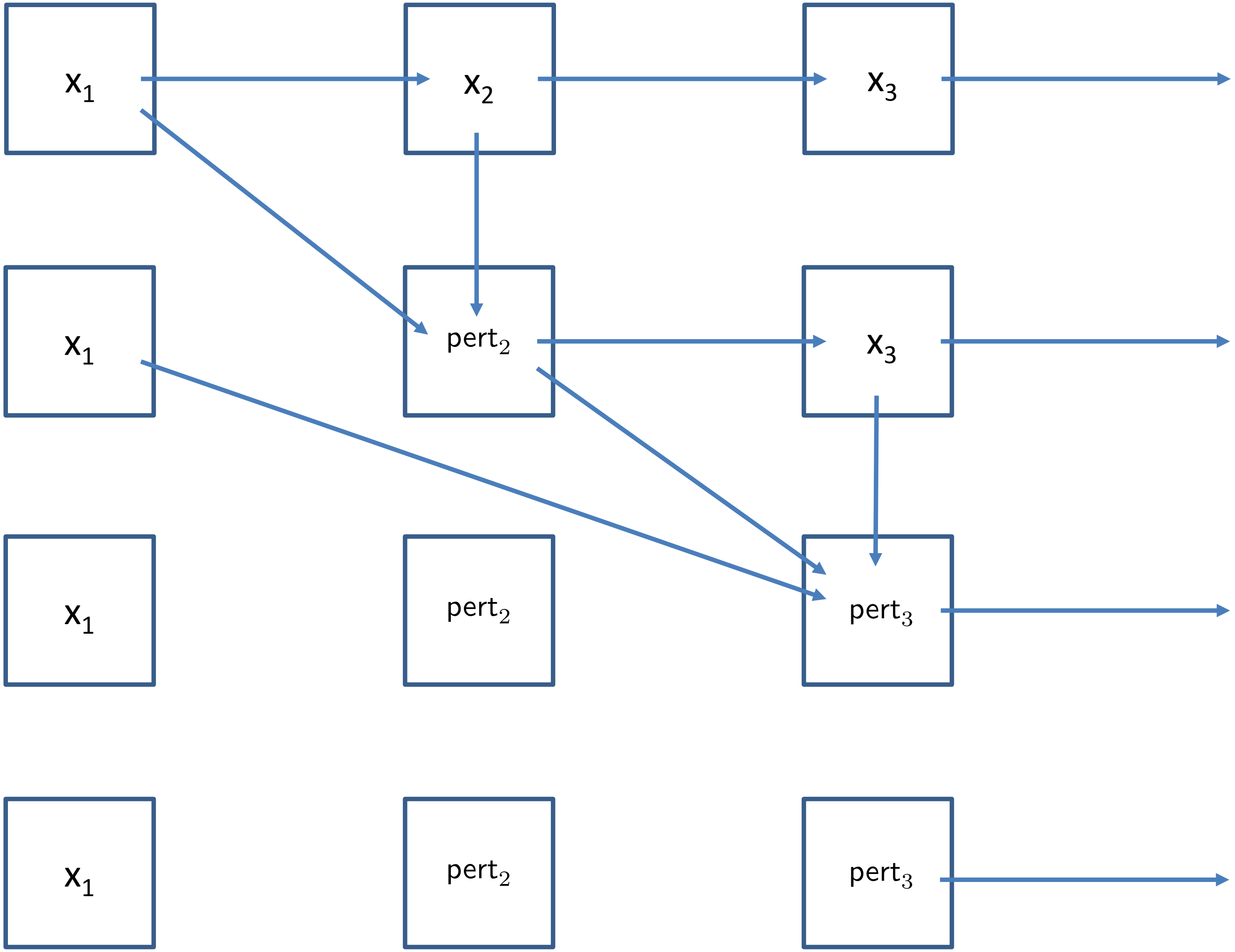
$$D_{i,j} = L_i^{-1} V_i \tilde{C}_{i,j} V_j^{-1} \qquad \forall i \in \{2, \dots, n\}, \forall j \in \{1, \dots, i-1\}$$

Nominal system matrix

Eigenvector matrices

$$[\tilde{C}_{i,j}]_{\ell,m} = [V_i^{-1} \underbrace{C_{i,i-1}}_{\text{Given coupling matrix for cascade}} D_{i-1,j} V_j]_{\ell,m} \underbrace{\left(1 - \frac{\lambda_{j,m}}{\lambda_{i,\ell}}\right)^{-1}}_{\text{The reason the disjoint spectrums assumption is needed}} \qquad \forall i \in \{2, \dots, n\}, \forall j \in \{1, \dots, i-1\}$$

Solutions of the cascade system



Linear Cascade

Nominal system with perturbed initial conditions

Asymptotic equivalence and zero asymptotic relative error

Asymptotic equivalence

$$\begin{aligned}\|\Pi_i \circ \mathsf{Lin}^{\circ t}(x_1,\dots,x_n) - \Pi_i \circ \mathsf{Nom}^{\circ t}(\mathsf{pert}(x_1,\dots,x_n))\| &\leq \sum_{j=1}^{i-1} \|D_{i,j}\| \|L_j^t \mathsf{pert}_j(x_1,\dots,x_j)\| \\ &\leq \left(\sum_{j=1}^{i-1} \|D_{i,j}\| \|\mathsf{pert}_j(x_1,\dots,x_j)\|\right) \|L_i\|^t\end{aligned}$$

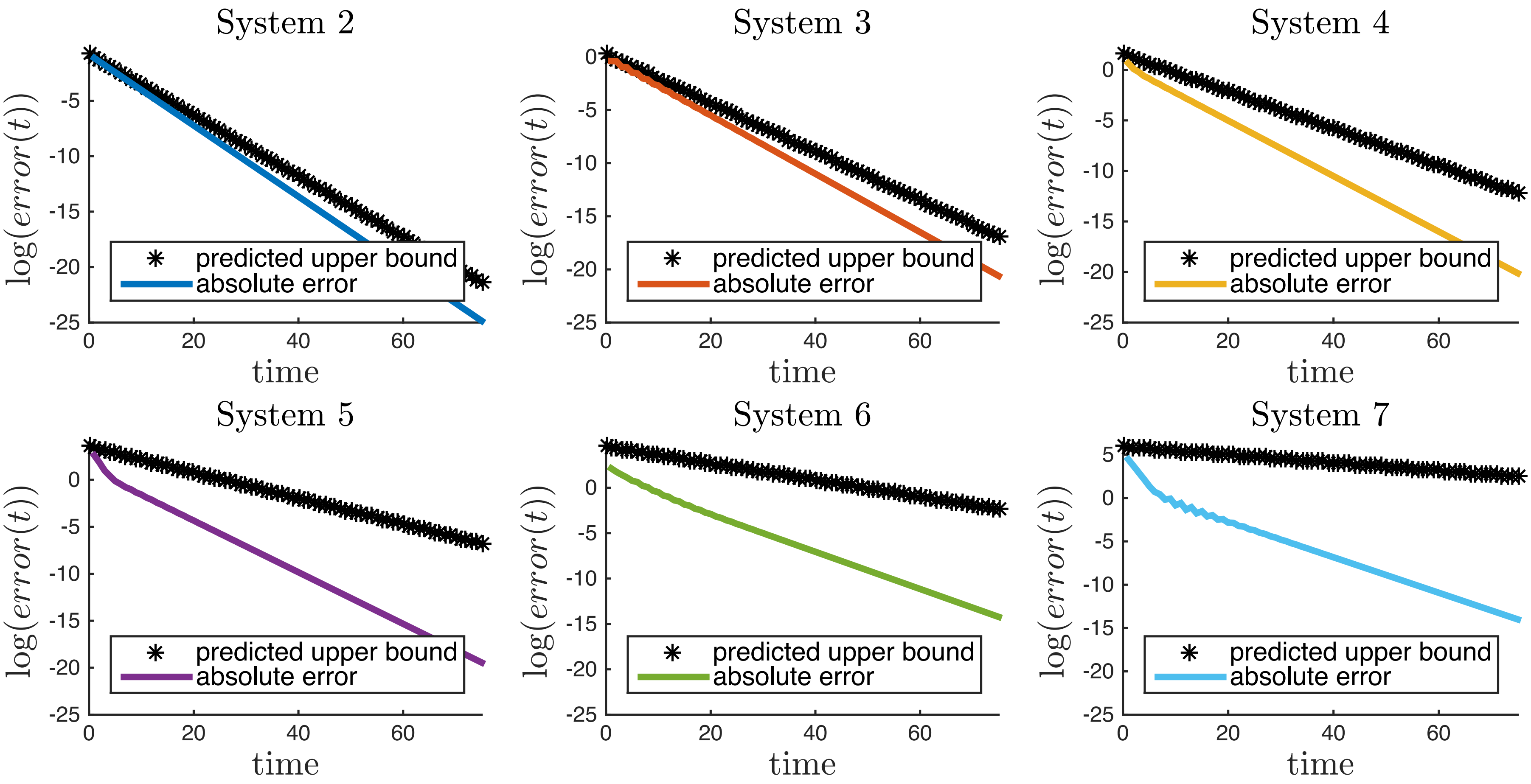
Zero asymptotic relative error

$$\lim_{t\to\infty} \frac{\|\Pi_i \circ \mathsf{Lin}^{\circ t}(x_1,\dots,x_n) - \Pi_i \circ \mathsf{Nom}^{\circ t}(\mathsf{pert}(x_1,\dots,x_n))\|}{\|L_i\|^t} = 0$$

Requires

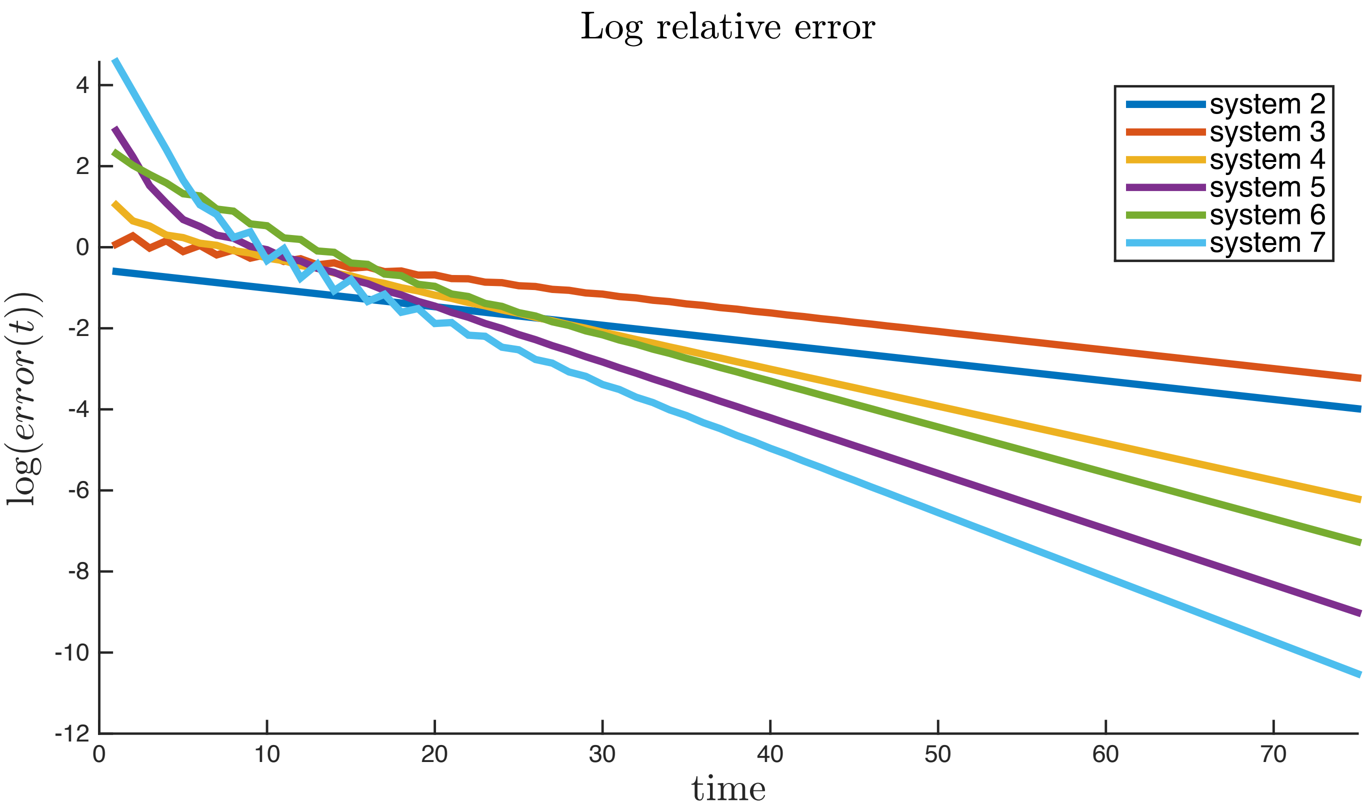
$$\|L_1\| < \|L_2\| < \cdots < \|L_n\|$$

Example: 7-layer cascade



Zero asymptotic relative error

$$\log\left(\frac{\|\Pi_i \circ \mathsf{Lin}^{\circ t}(x_1,\ldots,x_n) - \Pi_i \circ \mathsf{Nom}^{\circ t}(\mathsf{pert}(x_1,\ldots,x_n))\|}{\|L_i\|^t}\right)$$



Koopman operator and space of observables
Component and Cascade systems

$$x_i(t+1) = L_i x_i(t)$$

$$\mathcal{U}_{\text{Nom}_i} : \mathcal{A}_i \to \mathcal{A}_i$$

$$(\mathcal{U}^{\circ t}_{\text{Nom}_i} f)(x_i) = f(\text{Nom}_i^{\circ t}(x_i)) = f(L_i^t x_i)$$

▶ $$\psi_{i,s}(x_i) = (\hat{e}^*_{d_i,s} V_i^{-1}) x_i$$

A principal eigenfunction for ith system

Space of observables for linear cascade

$$\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$$

▶ $$(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) \equiv f_1(x_1) \cdots f_n(x_n)$$

Embed component principal eigenfunctions into tensor product

$$\psi_{(0,\ldots,0,s_i,0,\ldots,0)}(x_1, \ldots, x_n) = (\psi_{i,s_i} \circ \Pi_i)(x_1, \ldots, x_n) \equiv \psi_{i,s_i}(x_i)$$

$$\begin{aligned}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}(x_1, \ldots, x_n) &= \psi_{i,s_i}(x_i) \\ &\equiv (1 \otimes \cdots \otimes 1 \otimes \psi_{i,s_i} \otimes 1 \otimes \cdots \otimes 1)(x_1, \ldots, x_n)\end{aligned}$$

Asymptotic equivalence and relative error

Asymptotic equivalence

$$\left|\left(\mathcal{U}_{\mathsf{Lin}}^{\circ t}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)(x_1,\ldots,x_n) - \left(\mathcal{U}_{\mathsf{Nom}}^{\circ t}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)\circ \mathsf{pert}(x_1,\ldots,x_n)\right|$$

$$\leq \|\psi_{i,s_i}\| \sum_{j=1}^{i-1} \|D_{i,j}\| \|L_j^t \mathsf{pert}_j(x_1,\ldots,x_j)\|_{\mathbb{C}^{d_j}}.$$

Zero asymptotic relative error

$$\lim_{t\to\infty} \frac{\left|\left(\mathcal{U}_{\mathsf{Lin}}^{\circ t}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)(x_1,\ldots,x_n) - \left(\mathcal{U}_{\mathsf{Nom}}^{\circ t}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)\circ \mathsf{pert}(x_1,\ldots,x_n)\right|}{\|L_i\|^t} = 0$$

Preservation of principal eigenvalues and modified eigenfunctions

$\psi_{(0,\ldots,0,s_i,0,\ldots,0)} \circ \mathsf{pert}$ eigenfunction of $\mathcal{U}_{\mathsf{Lin}}$ at λ_{i,s_i}

Proof: (Peripheral eigenvalue case). Given

$$\lim_{t\to\infty} \frac{\left|\left(\mathcal{U}^{\circ t}_{\mathsf{Lin}}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)(x_1,\ldots,x_n) - \left(\mathcal{U}^{\circ t}_{\mathsf{Nom}}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)\circ \mathsf{pert}(x_1,\ldots,x_n)\right|}{\|L_i\|^t} = 0$$

$$\left(\mathcal{U}^{\circ t}_{\mathsf{Nom}}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\right)\circ \mathsf{pert}(x_1,\ldots,x_n) \;=\lambda^t_{i,s_i}\psi_{(0,\ldots,0,s_i,0,\ldots,0)}\circ \mathsf{pert}(x_1,\ldots,x_n)$$

Apply GLA theorem

$$\begin{aligned}\left\|\frac{1}{N}\sum_{t=0}^{N-1}\lambda^{-t}_{i,s_i}\mathcal{U}^{\circ t}_{\mathsf{Lin}}\psi_{s_i\hat{e}_{n,i}} - \psi_{s_i\hat{e}_{n,i}}\circ \mathsf{pert}\right\| &\le \frac{1}{N}\sum_{t=0}^{N-1}\left\|\lambda^{-t}_{i,s_i}\left(\mathcal{U}^{\circ t}_{\mathsf{Lin}}\psi_{s_i\hat{e}_{n,i}} - (\mathcal{U}^{\circ t}_{\mathsf{Nom}}\psi_{s_i\hat{e}_{n,i}})\circ \mathsf{pert}\right)\right\| \\ &= \frac{1}{N}\sum_{t=0}^{N-1}\frac{\left\|\left(\mathcal{U}^{\circ t}_{\mathsf{Lin}}\psi_{s_i\hat{e}_{n,i}} - (\mathcal{U}^{\circ t}_{\mathsf{Nom}}\psi_{s_i\hat{e}_{n,i}}\right)\circ \mathsf{pert}\right)\right\|}{\|L_i\|^t}.\end{aligned}$$

Nonlinear cascades

$$\begin{tikzcd}
\mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \arrow[r, "\mathsf{Lin}^{\circ t}"] \arrow[d, "\tau"] & \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \arrow[d, "\tau"] \\
\mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n} \arrow[r, "\mathsf{NonLin}^{\circ t}"] & \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_n}
\end{tikzcd}$$

- Topological conjugacy from linear to nonlinear cascade

$$\lim_{t\to\infty} \frac{\left|\mathcal{U}^{\circ t}_{\mathsf{NonLin}}(\psi_{(0,\ldots,0,s_i,0,\ldots,0)} \circ \tau^{-1})(\vec{y}) - \mathcal{U}^{\circ t}_{\tau\circ\mathsf{Nom}\circ\tau^{-1}}(\psi_{(0,\ldots,0,s_i,0,\ldots,0)} \circ \tau^{-1})((\tau \circ \mathsf{pert} \circ \tau^{-1})(\vec{y}))\right|}{\|L_i\|^t} = 0.$$

$$\underbrace{(\psi_{(0,\ldots,0,s_i,0,\ldots,0)} \circ \tau^{-1})}_{\text{Pullback of principal eigenfunction to (nominal) nonlinear system}}\underbrace{((\tau \circ \mathsf{pert} \circ \tau^{-1})}_{\text{Map of the linear perturbation function to nonlinear case}} \quad \text{eigenfunction of} \quad \mathcal{U}_{\mathsf{Nom}} \quad \text{at} \quad \lambda_{i,s_i}$$

Conclusions

- Under mild conditions that linear cascade is asymptotically equivalent to decoupled (nominal) system started from perturbed initial conditions

- The Koopman principal eigenvalues of component systems are also part of the spectrum for the linear cascade’s Koopman operator
- Principal eigenfunctions get modified by a composition with the perturbation functions

- The results extend to nonlinear cascades through topological conjugacy

Thank You

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