The convergence of research and innovation.
A toolbox for computing spectral properties of dynamical systems

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Outline

- Theory: “periodic approximation”
- Numerical method
- Examples
  - Cat map
  - Chirikov Standard map
- Conclusions
Theory: “periodic approximation”
The concept: “periodic approximation”

- Approximate the dynamical system by a periodic one.
- Notion originally introduced by Halmos.
- Katok & Stepin:
  - relate the rate of approximation of an automorphism by periodic approximation to the type of spectra the automorphism has

Our goal:
Use this concept to develop a numerical method to approximate the spectral decomposition of the Koopman operator.

Class of dynamical systems:
measure-preserving maps on compact domains.
Periodic approximation: Discretization

1. Partition domain into sets
   1. Elements of equal measure
   2. Diameter shrinks to zero
   3. Each partition is a refinement (subdivision) of the previous
2. Solve bipartite matching problem to get discrete map $T_n$
   1. Solution guaranteed by Hall’s marriage problem

\[
\mu(p_{n,i}) = \frac{\mu(X)}{q(n)}
\]
Periodic approximation: Hall Marriage Problem

Hall’s Marriage Problem

- Perfect matching iff for any subset of nodes $B$ on the left, its cardinality is less than the cardinality of the union of their neighbors, $N_G(B)$.

$$|B| \leq |N_G(B)|$$

$$|N_G(B)| := \sum_{k \in B} \left( \sum_{l: \mu(T(p_{n,k}) \cap p_{n,l} > 0)} 1 \right)$$

$$\geq \sum_{k \in B} \frac{q(n)}{\mu(X)} \sum_{l=1}^{q(n)} \mu(T(p_{n,k}) \cap p_{n,l})$$

$$= \sum_{k \in B} 1 = |B|$$

$T_n$ is a permutation $\Rightarrow$ unitary
Discretization overview (1)

continuous case:

\[ L^2(X, \mathcal{M}, \mu) := \{ g : X \to \mathbb{C} \mid \|g\| < \infty \}, \quad \|g\| := \left( \int_X |g(x)|^2 d\mu \right)^{\frac{1}{2}} \]

\[ \mathcal{U} : L^2(X, \mathcal{M}, \mu) \to L^2(X, \mathcal{M}, \mu) \]

\[ (\mathcal{U}g)(x) := g \circ T(x) \]

\[ \mathcal{U}g = \int_S e^{i\theta} dS_\theta g \]

\[ S_D g = \int_D dS_\theta g \]
Discretization overview (2)

Discretization of the automorphism:

\[ T_n : \mathcal{P}_n \mapsto \mathcal{P}_n \]

\( T_n \) is a periodic approximation

Discretization of the observable:

\[ (\mathcal{W}_n g)(x) = g_n(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{p_{n,j}}(x) \]

\[ g_{n,j} := \frac{q(n)}{\mu(X)} \int_X g(x) \chi_{p_{n,j}}(x) d\mu \]
Discretization overview (3)

\[ L^2_n(X, M, \mu) := \{ g_n : X \mapsto \mathbb{C} \mid \sum_{j=1}^{q(n)} c_j \chi_{p_n,j}(x), \quad c_j \in \mathbb{C} \} , \quad \chi_{p_n,j}(x) = \begin{cases} 1 & x \in p_n,j \\ 0 & x \notin p_n,j \end{cases} \]

\[ \mathcal{U}_n : L^2_n(X, M, \mu) \mapsto L^2_n(X, M, \mu) \]

\[ (\mathcal{U}_n g_n)(x) := \sum_{j=1}^{q(n)} g_n,j \chi_{T_n^{-1}(p_n,j)}(x) \]

\[ \mathcal{U}_n g_n = \sum_{k=1}^{q(n)} e^{i\theta_n,k} S_{n,\theta_n,k} g_n \]

\[ S_{n,D} g_n = \sum_{\theta_n,k \in D} S_{n,\theta_n,k} g_n \]

\[ \mathcal{U}_n v_{n,k} = e^{i\theta_n,k} v_{n,k} \]

\[ S_{n,\theta_n,k} g_n = v_{n,k} \langle v_{n,k}, g_n \rangle \]
Convergence of Periodic approximation

\[
\lim_{n \to \infty} \sum_{l=-k}^{k} d_H(T^l(A), T_n^l(A_n)) = 0
\]
Spectral convergence results

\[ \mathcal{U}^k g = \int_{\Omega} e^{ik\theta} dS_\theta g = \sum_{l=1}^{N} a_l e^{ik\theta_l} \phi_l + \int_{\Omega} e^{ik\theta} dS^*_\theta g, \quad k \in \mathbb{Z}. \]

(i) For some interval \( D \subset \mathbb{S} \), if \( g \in L^2(X, \mathcal{M}, \mu) \) has no nonzero modes \( a_k \) in (2) corresponding to eigenvalues on boundary \( \partial D = \overline{D} \setminus \text{int} D \), then:

\[ \lim_{n \to \infty} \| S_D g - S_{n,D} g_n \| = 0 \]

(ii) Define:

\[ \rho_{\alpha,n}(\theta; g_n) := \frac{\alpha}{2\pi} \sum_{j=1}^{\alpha} \| S_{n,D_{\alpha,j}} g_n \|^2 \chi_{D_{\alpha,j}}(\theta), \]

where

\[ D_{\alpha,j} := [\theta_{\alpha,j-1}, \theta_{\alpha,j}), \quad \theta_{\alpha,j} := -\pi + \frac{2\pi}{\alpha} j. \]

It follows that:

\[ \lim_{n,\alpha \to \infty} \int_{\mathbb{S}} \varphi(\theta) \rho_{\alpha,n}(\theta; g_n) d\theta = \int_{\mathbb{S}} \varphi(\theta) \rho(\theta; g) d\theta, \quad \text{for every } \varphi \in \mathcal{D}(\mathbb{S}). \]
Numerical method
Basic outline of numerical method

• Class of dynamical systems:
  
  \textit{Volume preserving maps on the d-torus}

• Two steps:
  
  1. Construct periodic approximation.
  
  2. Compute spectral projections of the periodic map.
1. Construct periodic approximation

Basic idea:
- Partition the d-torus into boxes and define grid points
- Evaluate map at grid points
- Construct neighborhood graph
- Solve bipartite matching problem
- Solution of bipartite matching is a candidate periodic discrete map.
2. Compute spectral projection

Basic idea:
- Use the Koopman operator of discrete map to approximate spectral projections.
- Discrete Koopman operator has a permutation structure.
- Find cycle decomposition of the permutation.
- Once the cycle decomposition is known, spectral projections can be computed with the FFT algorithm.

Matlab routine:
- Uses David Gleich MatlabBGL graph library to find periodic approximation.
- Code will be made public simultaneously with the papers.
Examples
Cat map

• Arnold’s cat map:

\[ T(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) \mod 1 \]

• An example of an Anosov diffeomorphism
• Has “Lebesgue spectrum”
• The following observables give the following densities:

\[
\begin{align*}
g_1(x_1, x_2) &= e^{2\pi i(2x_1 + x_2)} \\
g_2(x_1, x_2) &= e^{2\pi i(2x_1 + x_2)} + \frac{1}{2} e^{2\pi i(5x_1 + 3x_2)} \\
g_3(x_1, x_2) &= e^{2\pi i(2x_1 + x_2)} + \frac{1}{2} e^{2\pi i(5x_1 + 3x_2)} + \frac{1}{4} e^{2\pi i(13x_1 + 8x_2)}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \quad \rho(\theta; g_1) &= \frac{1}{2\pi} \\
\Rightarrow \quad \rho(\theta; g_2) &= \frac{1}{2\pi} \left( \frac{5}{4} + \cos \theta \right) \\
\Rightarrow \quad \rho(\theta; g_3) &= \frac{1}{2\pi} \left( \frac{21}{16} + \frac{10}{8} \cos \theta + \frac{1}{2} \cos 2\theta \right)
\end{align*}
\]
Cat map: some results

g_1(x_1, x_2) = \exp(i2\pi(2x_1 + x_2))
ρ(θ; g_1) = \frac{1}{2\pi}

n=250
Cat map: some results

\[ g_1(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) \]
\[ \rho(\theta; g_1) = \frac{1}{2\pi} \]

\( n=500 \)
Cat map: some results

\[ g_1(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) \]

\[ \rho(\theta; g_1) = \frac{1}{2\pi} \]

n=1000

[Diagram showing a cat map with marked angles and axes]
Cat map: some results

\[ g_1(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) \]

\[ \rho(\theta; g_1) = \frac{1}{2\pi} \]

n=2000
Cat map: some results

\[ g_2(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2)) \]

\[ \rho(\theta; g_2) = \frac{1}{2\pi} \left( \frac{5}{4} + \cos \theta \right) \]
Cat map: some results

\[ g_2(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2)) \]

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Image of a graph with the label 'n=1000'.
Cat map: some results

\[ g_2(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2)) \]

\[ \rho(\theta; g_2) = \frac{1}{2\pi} \left( \frac{5}{4} + \cos \theta \right) \]
Cat map: some results

\[ g_3(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2)) + \exp(i2\pi(13x_1 + 8x_2)) \]

\[ \rho(\theta; g_3) = \frac{1}{2\pi} \left( \frac{21}{16} + \frac{10}{8} \cos \theta + \frac{1}{2} \cos 2\theta \right) \]
Cat map: some results

\[ g_3(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2)) + \exp(i2\pi(13x_1 + 8x_2)) \]

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Chirikov Standard map

- Chirikov-Taylor map:

\[ T(x_1, x_2) = \left[ \begin{array}{l} x_1 + x_2 + K \sin(2\pi x_1) \\ x_2 + K \sin(2\pi x_1) \end{array} \right] \mod 1 \]

- Model of a kicked rotor.
- Has mixed spectra?
Chirikov Standard map: some results

$K=0.00$

$D = \left[ \frac{2\pi}{3} - 0.02, \frac{2\pi}{3} + 0.02 \right]$

$g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$
Chirikov Standard map: some results

\[ D = \left[ \frac{2\pi}{3} -0.02, \frac{2\pi}{3} + 0.02 \right] \]

\[ g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1 \]

\[ K=0.05 \]
Chirikov Standard map: some results

\[ D = \left[ \frac{2\pi}{3} -0.02, \frac{2\pi}{3} + 0.02 \right] \]

\[ g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1 \]

K=0.10
Chirikov Standard map: some results

\[ K = 0.15 \]

\[ D = \left[ \frac{2\pi}{3} - 0.02, \frac{2\pi}{3} + 0.02 \right] \]

\[ g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1 \]
Chirikov Standard map: some results

K=0.20

\[ D = \left[ 2\pi/3 -0.02, 2\pi/3 + 0.02 \right] \]

\[ g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1 \]
Chirikov Standard map: some results

$K=0.25$

$$D = [2\pi/3 -0.02, 2\pi/3 + 0.02]$$

$$g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$
Chirikov Standard map: some results

K = 0.30

\[ D = \left[ \frac{2\pi}{3} - 0.02, \frac{2\pi}{3} + 0.02 \right] \]

\[ g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1 \]
Chirikov Standard map: some results

$K = 0.35$

$$D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]$$

$$g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$
Conclusions
Conclusions

• Asymptotic convergence of the spectra is guaranteed in a weak-sense.
• Method can deal with continuous spectra.
• Method is only tractable for low dimensional maps.

Associated papers (in preparation):

- **Theory:** “A finite dimensional approximation of the Koopman operator with convergent spectral properties” N. Govindarajan, R Mohr, S. Chandrasekaran, I. Mezić.
- **Numerical method:** “A convergent numerical method for computing Koopman spectra of volume-preserving maps on the torus” N. Govindarajan, R. Mohr, S. Chandrasekaran, I. Mezić.
- **Generalization to flows:** “On the approximation of Koopman spectral properties of measure-preserving flows” N. Govindarajan, R. Mohr, S. Chandrasekaran, I. Mezić.