

A BACKSTEPPING CONTROLLER FOR A NONLINEAR PARTIAL DIFFERENTIAL EQUATION MODEL OF COMPRESSION SYSTEM INSTABILITIES*

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Abstract. We prove the existence and uniqueness of solutions in Sobolev spaces for the Moore–Greitzer nonlinear partial differential equation (PDE) model for compression system instabilities with mild conditions on the shape of the compressor characteristic and on the throttle control. To achieve this, the model is reformulated as an evolution equation on a Banach space. Using this new representation, we design a backstepping control of the model. Global stabilization of any axisymmetric equilibrium to the right of the peak of the compressor characteristic is achieved. We also prove that the dynamics can be restricted to the small neighborhood of the point on the left of the peak of the compressor characteristic. Thus, it is possible to restrict the magnitude of stall to arbitrary small values. In addition, finite-dimensional Galerkin projections of the partial differential equation model are studied. It is shown that truncated control laws stabilize truncated models. Numerical simulations of the model with and without control are presented.

Key words. control of partial differential equations, backstepping control of nonlinear differential equations, control of stall and surge in aeroengines, existence and uniqueness of solutions of partial differential equations

AMS subject classifications. 93C10, 93B29, 93B18, 53C65

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1. Introduction. Surge and stall instabilities that occur in compression systems of jet engines are the topic of much research effort these days for two reasons: efficiency and safety. In particular, jet engines are currently forced to operate in nonoptimal conditions (relatively large mass flow) in order to stay clear of the aforementioned instabilities. Surge is an oscillatory instability of the mean mass flow: upon the onset of surge the air in the compression system of a jet engine starts oscillating back and forth, thus severely impairing its performance. Stall is characterized by the appearance of the so-called *stall cells*—regions of decreased pressure rise and reversed mass flow—at isolated locations around the rim of the compressor. A simplified model of these instabilities has been proposed by Moore and Greitzer [15], and it is this model (sometimes called the *full Moore–Greitzer model*) that is the topic of the present paper. The model consists of a linear PDE governing the behavior of disturbances in the inlet region of the compression system, with nonlocal and nonlinear boundary conditions which describe the coupling of disturbances with the mean flow behavior. Since such a system is hard to analyze, most of the research has been directed toward establishing properties of low-order Galerkin truncations (see [15], [11]). The simplest

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approximate model used for bifurcation analysis (see [12]) and control (see [3], [6], [8], [9], [10], [16], [1]) called MG3 used only the first Fourier mode of the nonaxisymmetric flow disturbance (stall variable).

The Moore–Greitzer model deals with a simple compression system in which the air enters the compressor (with one or more rotor/stator stages), goes to a plenum, and exits through a throttle. Stationary operating points for the compressor correspond to a constant pressure rise across the compressor and a constant, circumferentially uniform mass flow through the compressor. The pressure rise versus mass flow curve representing the stationary operating points is called the compressor characteristic. For a given throttle opening the mass flow through the throttle is determined by the pressure drop across the throttle. The corresponding static relationship can be represented by the curve called the throttle characteristic. In a stationary condition the pressure rise across the compressor is balanced by the pressure drop across the throttle and the mass flow through the compressor and through the throttle are equal. Therefore, the intersection of the compressor characteristic and the throttle characteristic determines the operating point of the compressor. The operating point can be changed by adjusting the throttle opening.

The dynamic model for compression systems derived by Moore and Greitzer [15] describes the evolution of the mass flow and the pressure in plenum in a nonstationary condition. When the pressure rise across the compressor is not balanced by the pressure drop across the throttle, the resulting pressure difference is proportional to the rate of change of mass flow (i.e., mass acceleration). Similarly, if the mass flow through the compressor is not balanced by the mass flow through the throttle, the resulting difference is proportional to the rate of pressure rise in the plenum.

In terms of the dynamic model of a compression system, the stationary operating points are represented by the axisymmetric equilibria, surge corresponds to a limit cycle involving pressure rise and mass flow, while rotating stall is represented by a travelling wave of nonaxisymmetric mass flow around the compressor annulus with a constant low value of pressure rise.

From the point of view of efficiency, the desired operating points for the compressor should have high value of pressure rise and low mass flow that is uniform around the compressor annulus. However, the analysis of the dynamic model shows that the corresponding equilibria of the dynamic model have small domains of attraction that are shrinking as the pressure rise increases. A small disturbance is likely to force a transition of the compression system state to either rotating stall or surge.

In the present paper we study *stabilization* of a given axisymmetric equilibrium using the throttle opening as a control variable. The throttle opening is considered to be some function of the state of the system chosen so that the corresponding new dynamic model has a unique globally stable equilibrium at the prescribed location. While bounded disturbances would force the state of the system to evolve in some neighborhood of the desired equilibrium, after they disappear, the state would eventually return to an arbitrary small neighborhood of the equilibrium. Physically, the control would be implemented by varying the throttle opening in the way that prevents transition into rotating stall or surge.

After some manipulation, the dynamic model of compression system will be represented as an evolution equation in a Banach space. Stabilization of the desired equilibrium will be achieved by constructing a Lyapunov function for this equilibrium. A special pure feedback structure of the evolution equation will allow us to use *backstepping* for construction of a Lyapunov function. This technique has been intro-

duced in [7] and applied to many systems described by nonlinear ordinary differential equations (ODEs) with pure feedback structure, including a reduced-order model of compressor dynamics, MG3, in [8] and [1].

Experimental observations of the nonaxisymmetric flow disturbance (stall) behavior indicate that its shape is often far from sinusoidal [2]. Our investigations of the full Moore–Greitzer model show that the resulting nonlinear behavior of the compressor is well represented by the model [14]. It is then natural to ask what can be said about the control of stall and surge using the PDE model. In this paper we show that a global stabilization of the full Moore–Greitzer model is possible. We present a conceptually simple but not necessarily optimal way of constructing a globally stabilizing controller using backstepping control design [7]. To our knowledge, the present paper presents the first successful attempt to globally stabilize a nonlinear PDE using backstepping. We concentrate on a specific model here, but the methods that we develop can be used more broadly. A variety of evolution equation problems with a pure feedback structure can be treated in a way similar to that presented here. The backstepping method presents a powerful tool even in the context of PDEs.

The paper is organized as follows. In section 2 we introduce some notation and represent the full Moore–Greitzer model as an evolution equation in a Banach space, using an operator that is an infinite-dimensional version of that studied by Mansoux, Gyrling, Statiawan, and Paduaro in [11]. In section 3 we prove global existence and uniqueness of solutions of this evolution equation by a simple application of the contraction mapping principle. We present some a priori estimates which, together with the existence of a unique local solution, guarantee the existence of a unique global solution. In section 4 we design a backstepping controller for the full Moore–Greitzer model. We show that the peak and any axisymmetric equilibrium to the right of the peak can be globally asymptotically stabilized.

In the case when the set-point parameter in the controller is such that there is no stable axisymmetric equilibrium we can still guarantee that the dynamics of the closed-loop system are confined to a ball, whose radius can be made arbitrarily small by choosing sufficiently high gains in the controller.

In section 5 we prove that the truncated feedback controller globally stabilizes the system of $2n + 2$ ODEs consisting of the Galerkin projection of the PDE describing the stall dynamics onto its first n -modes and the two ODEs describing the surge dynamics. The results are valid for a general compressor characteristic.

2. Preliminaries.

2.1. The Moore–Greitzer model. The full Moore–Greitzer model is described by the following equations (cf. [15])

$$(1) \quad l_c \frac{d\Phi}{d\xi} = -\Psi(\xi) + \frac{1}{2\pi} \int_0^{2\pi} \Psi_c(\Phi + \phi'_\eta|_{\eta=0}) d\theta,$$

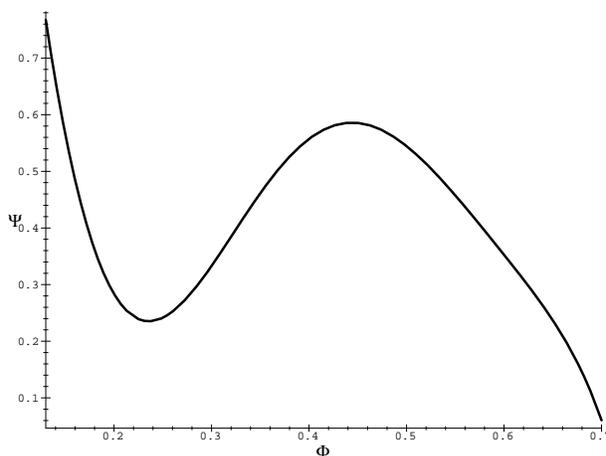
$$(2) \quad l_c \frac{d\Psi}{d\xi} = \frac{1}{4B^2} (\Phi(\xi) - K_T(\Psi, u)),$$

where ϕ' solves Laplace’s equation

$$(3) \quad \phi'_{\eta\eta} + \phi'_{\theta\theta} = 0$$

for $(\eta, \theta) \in [0, 2\pi] \times (-\infty, 0)$. The boundary conditions are periodic in θ ,

$$(4) \quad \frac{\partial}{\partial \xi} (m\phi' + \frac{1}{a}\phi'_\eta) - \left(\Psi_c(\Phi + \phi'_\eta) - \frac{1}{2\pi} \int_0^{2\pi} \Psi_c(\Phi + \phi'_\eta) d\theta - \frac{1}{2a}\phi'_{\theta\eta} \right) = 0$$

FIG. 1. C'_3 compressor characteristic.

at $\eta = 0$. At $\eta = -\infty$ we have

$$(5) \quad \phi' = 0.$$

(Note that we try to keep our notation consistent with that of the original paper by Moore and Greitzer [15]. In particular, following [15] we use ξ to denote a nondimensional time variable and η to denote a nondimensional axial distance variable.) The state variables of this model are Φ , the nondimensionalized annulus averaged mass flow coefficient through the compressor; Ψ , the nondimensionalized annulus averaged pressure rise coefficient across the compressor; and ϕ' , the disturbance velocity potential. (Note that the prime symbol in ϕ' *does not* refer to differentiation.) The function $\Psi_c(\phi)$ is called the compressor characteristic and is found empirically. It gives the local pressure rise when the local mass flow is ϕ . For most compressors it has an S shape as seen in Figure 1. The parameters a , m , l_c , and B are determined by the geometry of the compressor and the throttle parameter $K_T(\Psi, u)$ is the fraction of the throttle opening. Since the throttle parameter can be varied, it will be used as the control. We assume that we can modify $K_T(\Psi, u)$ at will by a choice of the control function u .

We assume that the compressor characteristic $\Psi_c(\Phi)$ is a general S-shaped curve. In particular, we assume that the following characteristics hold.

1. The characteristic $\Psi_c(\Phi)$ is twice continuously differentiable.
2. The characteristic has one peak (Φ_0, Ψ_0) and, to the left of the peak, one well. The characteristic is strictly decreasing to the right of the peak and to the left of the well; it is strictly increasing between the well and the peak.
3. The characteristic has exactly one inflection point $(\Phi_{infl}, \Psi_{infl})$ between the well and the peak. One has $\Psi_c''(\Phi) < 0$ for $\Phi > \Phi_{infl}$, and $\Psi_c''(\Phi)$ is bounded away from zero on any interval $[\Gamma, +\infty)$ for $\Gamma > \Phi_{infl}$.

Figure 1 shows a typical compressor characteristic $\Psi_c(\phi)$.

Let us give here a physical interpretation for the shape of the compressor characteristic. The desired possible stationary operating points of the compressor lie on the

decreasing part of the characteristic to the right of the peak. The lower the value of the axial component of the mass flow entering compressor, the more flow turning is achieved by the blades. Consequently, more work is done on the air by the compressor blades and the pressure rise is higher. However, there is a limit to the value of pressure rise that can be achieved. When the axial component of the incoming air velocity is small relative to the rotor velocity, so that the air is approaching the blade at a high angle of attack, the flow separates on the suction side of a compressor blade; i.e., the blade *stalls*. When a blade stalls, the pressure rise at that blade drops significantly. The pressure rise of a stalled blade is represented by the part of the characteristic between the well and the peak. The peak represents the *stall inception point*: the maximum pressure rise is obtained by a blade that is just about to stall. When the flow is reversed, the air at the suction side of the blade is attached again and the pressure starts to rise. This is represented by the part of the characteristic to the left of the well, called the back-flow part of the characteristic.

The physical mechanism for rotating stall and surge inception can now be explained. When a blade stalls, the pressure in the plenum is usually greater than the local pressure rise produced by the compressor, so that the incoming air at the stalled blade faces a negative pressure gradient and hence has a negative acceleration. The mass flow at the stalled blade passage is locally reduced. There are several possible scenarios of how the situation will evolve. The extreme ones are a transition to a surge or rotating stall condition.

In surge the pressure in the plenum does not drop fast enough, and all the blades stall at the same time. The flow eventually reverses, as the only mechanism to balance the high pressure in the plenum at the stalled blades is for the mass flow to reach the back-flow part of the characteristic. This transition from the neighborhood of the stall inception point to the back-flow characteristic is very fast. The pressure in the plenum is now dropping, as the air escapes from the plenum both through the throttle and through the compressor. The pressure in the plenum eventually drops below the value of the pressure rise on the back-flow part of the characteristic and the pressure gradient becomes positive. The air accelerates slowly until zero mass flow is reached. The plenum pressure is now below the well value. The compressor starts to deliver more pressure rise while the pressure in plenum is about the well level, so the mass flow accelerates fast. Past the value corresponding to the peak the flow at the blades becomes attached. When the flow through compressor becomes bigger than the flow through the throttle, the pressure starts to rise and becomes bigger than the one produced by the compressor. The flow starts to decrease. When the flow reaches the value corresponding to the peak, pressure in the plenum is about the peak value, which is the condition of a stall inception. One full surge cycle is now completed and the next one is about to start.

When rotating stall occurs, one or several blades stall. Locally, the flow is redirected to the neighboring unstalled blades. On one side of the region of stalled blades, the angle of attack of the air flow will increase, causing more blades to stall. On the other side the angle of attack will decrease, making the blades on that side less susceptible to stalling. The air is coming to the blades in the direction of spinning rotor at high angle of attack. These blades are likely to stall. At the same time the blades neighboring the stalled ones in the direction opposite to the spinning rotor accept air at the lower angle of attack than the stalled ones so they are not likely to stall. More air coming through these blades may lower the angle of attack on the first stalled blade, which will result in more local pressure rise. If the pressure in plenum

drops fast enough, the pressure gradient on some of the stalled blades will become positive and these blades will unstall. A stable rotating stall condition may develop when some of the blades operate in a stalled condition and the rest are not stalled. The cells of stalled air travel around the compressor so that each blade periodically becomes stalled and unstalled. Note that in a rotating stall condition the average pressure rise delivered by the compressor is low, as the stalled blades barely do any work on the air. Note also that in a stable rotating stall condition the air mass flow in the stalled blade passages is reversed, as the stalled blades cannot deliver enough pressure rise to balance the pressure in the plenum.

At this point the physical mechanism for the stabilization of the operating point close to the peak of the characteristic by varying the throttle opening can be explained. In both stall and surge inception the mechanism of instability is the same: a stalled blade cannot deliver enough pressure rise to balance the high pressure in the plenum and the resulting negative pressure gradient decelerates and eventually reverses the flow at some (stall) or all (surge) blade passages. The control action basically amounts to opening the throttle fast enough so that the pressure in the plenum drops faster than the pressure at a stalled blade. This produces the positive pressure gradient that accelerates the flow to the desired value. After this value is reached, the throttle is closed again.

2.2. Some function spaces. Let \mathcal{L}^2 be the space of square integrable functions on the circle $[0, 2\pi]$ and denote the norm by $\|\cdot\|_{\mathcal{L}^2}$. Let L^2 , L^∞ , and H^k , for $k = 1, 2, \dots$, denote the subspaces of \mathcal{L}^2 with zero average and norms $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{H^k}$. These norms are given by

$$\begin{aligned} \|g\|_{L^2} &:= \left(\int_0^{2\pi} g^2 d\theta\right)^{\frac{1}{2}} = \left(\pi \sum_{p=1}^\infty A_p(\xi)^2\right)^{\frac{1}{2}}, \\ \|g\|_{H^k} &:= \left(\int_0^{2\pi} \left(\frac{\partial^k g}{\partial \theta^k}\right)^2 d\theta\right)^{\frac{1}{2}} = \left(\pi \sum_{p=1}^\infty (p^k A_p(\xi))^2\right)^{\frac{1}{2}}, \\ \|g\|_{L^\infty} &:= \operatorname{esssup}_{\theta \in [0, 2\pi]} g. \end{aligned}$$

Here, the A_p 's represent the magnitudes of the complex Fourier coefficients of g ,

$$g = \sum_{p=1}^\infty A_p(\xi) \sin(p\theta + r_p(\xi)).$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product of L^2 :

$$\langle g_1, g_2 \rangle = \int_0^{2\pi} g_1 g_2 d\theta = \pi \sum_{p=1}^\infty A_p B_p.$$

Here, the A_p 's and the B_p 's represent the Fourier coefficients of the functions g_1 and g_2 , respectively.

Let C^0 denote the space of continuous functions on the circle $[0, 2\pi]$ with zero average, with the norm

$$\|g\|_{C^0} := \max_{\theta \in [0, 2\pi]} g.$$

Note that for $g \in C^0$ one has $\|g\|_{C^0} = \|g\|_{L^\infty}$. Thus, to avoid using too many symbols we will use $\|g\|_{L^\infty}$ to denote the norm of C^0 functions.

For future reference, we collect here some inequalities in the following lemma.

LEMMA 2.1. *One has $H^1 \hookrightarrow C^0 \hookrightarrow L^\infty \hookrightarrow L^2$ and*

$$\begin{aligned} \|g\|_{L^2} &\leq \sqrt{2\pi} \|g\|_{L^\infty}, \\ \|g\|_{L^\infty} &\leq \frac{\sqrt{\pi}}{\sqrt{6}} \|g\|_{H^1}, \\ \|g\|_{L^2} &\leq \|g\|_{H^1}. \end{aligned}$$

Proof. We first prove the inequalities. The first inequality is clear. The second one follows from $\|g\|_{L^\infty} \leq \sum_{p=1}^\infty |A_p(\xi)| \leq (\sum_{p=1}^\infty (\frac{1}{p^2}))^{\frac{1}{2}} (\sum_{p=1}^\infty (pA_p(\xi))^2)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{\sqrt{6}} \|g\|_{H^1}$. The third one is the Poincaré inequality. $H^1 \hookrightarrow C^0$ is the Sobolev embedding of H^1 into C^0 in spatial dimension one. The other embeddings follow from the inequalities. \square

Assume for now that ϕ' can be represented as

$$(6) \quad \phi' = \sum_{p=1}^\infty e^{pn} \alpha_p(\xi) \sin(p\theta + r_p(\xi)),$$

where α_p and r_p are real functions.

Let $g := \phi'_\eta|_{\eta=0}$; then

$$g = \sum_{p=1}^\infty p\alpha_p(\xi) \sin(p\theta + r_p(\xi)) =: \sum_{p=1}^\infty A_p(\xi) \sin(p\theta + r_p(\xi)).$$

Equation (4) can therefore be written as

$$(7) \quad \frac{\partial}{\partial \xi} Kg = a \left(\Psi_c(\Phi + g) - \frac{1}{2\pi} \int_0^{2\pi} \Psi_c(\Phi + g) d\theta - \frac{1}{2} \frac{\partial g}{\partial \theta} \right),$$

where the operator K is defined as follows:

$$(8) \quad K \sum_{p=1}^\infty \alpha_p(\xi) \sin(p\theta + r_p(\xi)) = \sum_{p=1}^\infty \left(1 + \frac{am}{p} \right) \alpha_p(\xi) \sin(p\theta + r_p(\xi)).$$

The operator K is an infinite-dimensional analogue of the operator introduced in [11] for a study of finite-dimensional truncations of the full Moore–Greitzer model.

Remark 2.1. Suppose that we can show that the system (1), (2), and (7) has a unique solution such that $g \in H^1$. Then from the Fourier series representation of g we can calculate the corresponding potential ϕ' . Since $g \in H^1$, it follows that on the cylinder $[0, 2\pi] \times (-\infty, 0)$ the potential ϕ' is in the Sobolev space $H^2([0, 2\pi] \times (-\infty, 0))$. In particular, the partial derivatives $\phi'_{\eta\eta}$ and $\phi'_{\theta\theta}$ are in $L^2([0, 2\pi] \times (-\infty, 0))$. Note also that from (6) it follows that ϕ' satisfies the Laplace equation (3) and the boundary conditions (4) and (5). The existence of solutions of the full Moore–Greitzer model follows. The uniqueness follows from the uniqueness of g . It therefore suffices to prove the existence and uniqueness of solutions for the system (1), (2), and (7) to obtain the existence and uniqueness of solutions of the full Moore–Greitzer model.

The variable g represents the nonaxisymmetric mass flow disturbance, i.e., the stall. We shall refer to g as the *stall variable*.

PROPOSITION 2.1. *Let $Z = L^2$ or H^k , for $k = 1, 2, \dots$. $K : Z \mapsto Z$ is a bounded, self-adjoint, positive definite operator with a bounded inverse. One has $\|K\|_Z = 1 + am$, and $\|K^{-1}\|_Z = 1$. Moreover, K , $\frac{\partial}{\partial \xi}$, and $\frac{\partial}{\partial \theta}$ commute.*

Proof. We have $\|Kg\|_Z \leq (1 + \frac{am}{1})\|g\|_Z$ so K is bounded with $\|K\|_Z = 1 + am$. Similarly, K^{-1} is bounded with $\|K^{-1}\|_Z = 1$. Furthermore, $\langle g, Kg \rangle \geq \|g\|_Z$, so K is positive definite and bounded away from zero. It is easy to see that K is self-adjoint.

On their domains the operators $\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial \theta}$, and K all commute. This is clear by letting them operate termwise on the Fourier series. \square

For future reference, we note that the inverse of K is

$$K^{-1} \left(\sum_{p=1}^{\infty} A_p(\xi) \sin(p\theta + r_p(\xi)) \right) := \sum_{p=1}^{\infty} \left(\frac{p}{p + am} \right) A_p(\xi) (\sin(p\theta + r_p(\xi))).$$

Using K we can define weighted L^2 and H^k norms as follows:

$$\begin{aligned} \|g\|_{L_K^2} &:= \sqrt{\langle g, Kg \rangle}, \\ \|g\|_{H_K^k} &:= \sqrt{\langle \frac{\partial^k g}{\partial \theta^k}, K \frac{\partial^k g}{\partial \theta^k} \rangle}. \end{aligned}$$

Note that L^2 and H^k norms are equivalent with their weighted counterparts. In fact, one has Lemma 2.2.

LEMMA 2.2. *We have that*

$$\begin{aligned} \|g\|_{L^2} &\leq \|g\|_{L_K^2} \leq \sqrt{1 + am} \|g\|_{L^2}, \\ \|g\|_{H^k} &\leq \|g\|_{H_K^k} \leq \sqrt{1 + am} \|g\|_{H^k}. \end{aligned}$$

We define

$$\overline{\Psi}_c := \frac{1}{2\pi} \int_0^{2\pi} \Psi_c(\Phi + g(\xi, \theta)) d\theta.$$

Then we can rewrite the model (4), (1), and (2) as

$$(9) \quad \frac{\partial}{\partial \xi} g(\xi, \theta) = K^{-1} \left(a(\Psi_c(\Phi(\xi) + g(\xi, \theta)) - \overline{\Psi}_c) - \frac{1}{2} \frac{\partial g(\xi, \theta)}{\partial \theta} \right),$$

$$(10) \quad \frac{d\Phi}{d\xi} = \frac{1}{l_c} (\overline{\Psi}_c - \Psi(\xi)),$$

$$(11) \quad \frac{d\Psi}{d\xi} = \frac{1}{4l_c B^2} (\Phi(\xi) - K_T(\Psi, u)).$$

We will frequently use a formula for a difference of values of a C^1 function at two points.

LEMMA 2.3. *Let f be a C^1 function. Then*

$$f(x + \Delta x) - f(x) = \left(\int_0^1 f'(x + s\Delta x) ds \right) \Delta x.$$

3. Existence and uniqueness of solutions.

3.1. Moore–Greitzer model as an evolution equation on a Banach space.

To prove the existence and uniqueness of solutions of the full Moore–Greitzer model we represent the model as an evolution equation of the form

$$(12) \quad \frac{dx}{d\xi} = Ax + f(x),$$

where x belongs to a Banach space X , A is an unbounded operator in X , and f is a nonlinear operator.

Let X be a Banach space. We define the following two spaces:

$$C(0, T; X) := \{x(\cdot) : [0, T] \rightarrow X \text{ is strongly continuous in } \|\cdot\|_X \text{ norm}\},$$

$$C^1(0, T; X) := \{x(\cdot) : [0, T] \rightarrow X \text{ is continuously differentiable in } \|\cdot\|_X \text{ norm}\}$$

with corresponding norms

$$\|x\|_{C(0,T;X)} = \sup_{\xi \in [0,T]} \|x(\xi)\|_X,$$

$$\|x\|_{C^1(0,T;X)} = \sup_{\xi \in [0,T]} \|x(\xi)\|_X + \sup_{\xi \in [0,T]} \left\| \frac{d}{d\xi} x(\xi) \right\|_X.$$

We are going to use the following corollary from Kato’s theorem [4], [5].

THEOREM 3.1. *Let X be a Banach space, and let A be a generator of a strongly continuous semigroup on X . Let Y be the domain of A . Suppose that $f(\cdot)$ satisfies the conditions*

$$(13) \quad \|f(x)\|_Y \leq C_{bdd}(\|x\|_Y)$$

and

$$(14) \quad \|f(x_1) - f(x_2)\|_X \leq C_{Lip}(\|x_1\|_X, \|x_2\|_X, \|Ax_1\|_X, \|Ax_2\|_X) \|x_1 - x_2\|_X,$$

where functions C_{bdd} and C_{Lip} are bounded on bounded sets. Then for all $x_0 \in Y$ there exists a unique local strong solution of

$$(15) \quad \frac{dx}{d\xi} = Ax + f(x)$$

such that

$$(16) \quad x \in C(0, \delta; Y) \cap C^1(0, \delta; X), \quad x(0) = x_0$$

for some $\delta > 0$.

Proof. See Theorem 10 of [4]. \square

Define, for $k = 0, 1, \dots$, the spaces

$$X^k := H_K^k \times \mathbf{R}^2$$

(with $H_K^0 := L_K^2$). The norms on X^k are defined by

$$\|(g, \Phi, \Psi)\|_{X^k}^2 = \|g\|_{H_K^k}^2 + |\Phi|^2 + |\Psi|^2.$$

In this paper we will apply Theorem 3.1 with $X = X^{k-1}$ and $Y = X^k$.

3.2. Local existence and uniqueness. With suitable conditions on the compressor characteristic, Ψ_c , the local existence of X^k solutions becomes rather elementary. In an attempt to appeal to a larger audience, here we will present a detailed proof of the local existence and uniqueness of X^1 solutions. We then state a theorem which gives local existence and uniqueness in X^k and outline the proof.

We are going to apply Theorem 3.1 with

$$X = X^0 = L_K^2 \times \mathbf{R}^2, \quad Y = X^1 = H_K^1 \times \mathbf{R}^2,$$

with the norms

$$\begin{aligned} \|(g, \Phi, \Psi)\|_{X^1}^2 &= \|g\|_{H_K^1}^2 + |\Phi|^2 + |\Psi|^2, \\ \|(g, \Phi, \Psi)\|_{X^0}^2 &= \|g\|_{L_K^2}^2 + |\Phi|^2 + |\Psi|^2. \end{aligned}$$

These spaces are both Hilbert spaces, X^1 is continuously embedded in X^0 , and X^1 is dense in X^0 . We now define the operator $A : X^1 \rightarrow X^0$ as follows:

$$(17) \quad A(g, \Phi, \Psi) := \left(-\frac{1}{2}K^{-1}\frac{\partial g}{\partial \theta}, 0, 0 \right).$$

This operator is closed and, as we will show, it is an infinitesimal generator of a strongly continuous unitary semigroup on X^0 . We define

$$(18) \quad f(g, \Phi, \Psi) := (aK^{-1}(\Psi_c(\Phi + g) - \bar{\Psi}_c), \frac{1}{l_c}(\bar{\Psi}_c - \Psi), \frac{1}{4l_c B^2}(\Phi - K_T(\Psi, u))).$$

Remark 3.1. Using a square throttle characteristic and constant throttle control u (cf. [15]) will cause K_T not to be Lipschitz on the hyperplane defined by $\Psi = 0$. However, if we use a feedback control of the form $u = u(g, \Phi, \Psi)$, $K_T(\Psi, u)$ (and hence also $f(g, \Phi, \Psi)$) becomes a function of all the state variables. We assume that the feedback was chosen such that $K_T(\Psi, u)$ is Lipschitz on bounded subsets of \mathbf{R} .

Having defined the function spaces, the operator A , and the nonlinear operator f , we can prove the local existence and uniqueness of solutions of the full Moore–Greitzer model. For this, we will show in the following two lemmas that A given by (17) and $f(g, \Phi, \Psi)$ given by (18) satisfy the conditions of Theorem 3.1.

LEMMA 3.1. *The operator A given by (17) is a generator of a strongly continuous unitary semigroup on X^0 .*

Proof. Using the fact that K is self-adjoint and integration by parts one can prove that $A^* = -A$; i.e., A is a skew-adjoint operator. Thus, A generates a strongly continuous unitary semigroup on X^0 (cf. Theorem 8 of [4]). \square

LEMMA 3.2. *Suppose that $\Psi_c \in C^1(\mathbf{R})$. We also assume that $K_T(\Psi, u)$ is bounded from X^1 to \mathbf{R} and X^0 -Lipschitz on X^1 -bounded sets, i.e., for all $x_i = (g_i, \Phi_i, \Psi_i) \in X^1$, $i = 1, 2$, $K_T(\Psi, u)$ satisfies*

$$|K_T(\Psi_1, u(x_1)) - K_T(\Psi_2, u(x_2))| \leq C_K \|x_1 - x_2\|_{X^0},$$

where C_K is a function of X^1 norms of x_i , $i = 1, 2$, which is bounded on bounded sets in X^1 . Then the function $f(g, \Phi, \Psi)$ given by (18) satisfies conditions (13) and (14) of Theorem 3.1.

Proof. Let \mathcal{M} be a bounded subset of X^1 and let $(g, \Phi, \Psi) \in \mathcal{M}$ be arbitrary. Then, because of embedding $H^1 \hookrightarrow C^0$, for every θ , $\Phi + g(\theta)$ belongs to a

bounded interval $I_{\mathcal{M}} \subset \mathbf{R}$. (Note that $I_{\mathcal{M}}$ depends only on \mathcal{M} .) Thus, $|\Psi'_c(\Phi + g)| \leq \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)|$. Therefore,

$$\begin{aligned} \|aK^{-1}(\Psi_c(\Phi + g) - \bar{\Psi}_c)\|_{H_K^1}^2 &\leq a^2 \|K^{-1}\| \left\| \frac{\partial}{\partial \theta} \Psi_c(\Phi + g) \right\|_{L^2}^2 \\ &\leq a^2 \|K^{-1}\| \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)|^2 \|g\|_{H^1}^2 \\ (19) \qquad &\leq a^2 \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)|^2 \|g\|_{H_K^1}^2. \end{aligned}$$

We also have

$$(20) \qquad |\bar{\Psi}_c| = \left| \int_0^{2\pi} \Psi_c(\Phi + g) d\theta \right| \leq \sup_{\phi \in I_{\mathcal{M}}} \Psi_c(\phi).$$

Using (19) and (20), we easily show that $f(g, \Phi, \Psi)$ satisfies (13).

To show that $f(g, \Phi, \Psi)$ satisfies (14), let $x_1 = (g_1, \Phi_1, \Psi_1) \in \mathcal{M}$ and $x_2 = (g_2, \Phi_2, \Psi_2) \in \mathcal{M}$ be arbitrary. To simplify notation, let us denote $F_i := \Psi_c(\Phi_i + g_i)$ for $i = 1, 2$. Recall that $\overline{F_i}$ denotes the average value of F_i . We have

$$\begin{aligned} &\|f(x_1) - f(x_2)\|_{X^0}^2 \\ &= a^2 \langle K^{-1}((F_1 - F_2) - \overline{(F_1 - F_2)}), (F_1 - F_2) - \overline{(F_1 - F_2)} \rangle \\ &\quad + \frac{1}{l_c^2} |\overline{(F_1 - F_2)} - (\Psi_1 - \Psi_2)|^2 \\ &\quad + \frac{1}{(4l_c B^2)^2} |(\Phi_1 - \Phi_2) - (K_T(\Psi_1, u(x_1)) - K_T(\Psi_2, u(x_2)))|^2 \\ &\leq a^2 \|K^{-1}\|_{L^2} \|(F_1 - F_2) - \overline{(F_1 - F_2)}\|_{L^2}^2 \\ &\quad + \frac{2}{l_c^2} |\overline{(F_1 - F_2)}|^2 + \frac{2}{l_c^2} |(\Psi_1 - \Psi_2)|^2 \\ &\quad + \frac{2}{(4l_c B^2)^2} |(\Phi_1 - \Phi_2)|^2 + \frac{2C_K^2}{(4l_c B^2)^2} \|x_1 - x_2\|_{X^0}^2. \end{aligned}$$

Note that $\|K^{-1}\|_{L^2} = 1$ and

$$\|F_1 - F_2\|_{L^2}^2 = \|(F_1 - F_2) - \overline{(F_1 - F_2)}\|_{L^2}^2 + 2\pi |\overline{(F_1 - F_2)}|^2.$$

Hence,

$$\begin{aligned} &\|f(x_1) - f(x_2)\|_{X^0}^2 \\ &\leq C_1 \|F_1 - F_2\|_{L^2}^2 + C_2 \|x_1 - x_2\|_{X^0}^2, \end{aligned}$$

where $C_1 := a^2 + \frac{2}{l_c^2}$ and $C_2 := \frac{2}{l_c^2} + \frac{2}{(4l_c B^2)^2} + \frac{2C_K^2}{(4l_c B^2)^2}$. We will show that

$$(21) \qquad \|(F_1 - F_2)\|_{L^2}^2 \leq C_3 \|x_1 - x_2\|_{X^0}^2,$$

where C_3 is a function of X^1 norms of (g_i, Φ_i, Ψ_i) , $i = 1, 2$, which is bounded on bounded sets in X^1 . For this, note that by Lemma 2.3

$$\|F_1 - F_2\|_{L^2}^2 = \left\| \left(\int_0^1 \Psi'_c(\Phi_2 + g_2 + s(\Phi_1 + g_1 - \Phi_2 - g_2)) ds \right) (\Phi_1 - \Phi_2 + g_1 - g_2) \right\|_{L^2}^2$$

$$\begin{aligned} &\leq 2 \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)|^2 (|\Phi_1 - \Phi_2|^2 + \|g_1 - g_2\|_{L^2_K}^2) \\ &\leq 2 \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)|^2 \|x_1 - x_2\|_{X^0}^2. \end{aligned}$$

Therefore, (21) holds with $C_3 := 2 \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)|^2$. Note that C_3 is bounded on bounded sets in X^1 . Thus,

$$\|f(x_1) - f(x_2)\|_{X^0}^2 \leq C_4 \|x_1 - x_2\|_{X^0}^2,$$

where $C_4 := C_1 C_3 + C_2$. Note that C_4 is bounded on bounded sets in X^1 . Therefore, $f(g, \Phi, \Psi)$ satisfies (14). \square

Therefore, we can state the following result.

THEOREM 3.2. *Assume that Ψ_c is a C^1 function. Then the Cauchy problem*

$$(22) \quad \frac{dx}{d\xi} = Ax + f(x, u), \quad x(0) = x_0 \in X^1$$

has a unique solution $x \in C(0, \delta; X^1) \cap C^1(0, \delta; X^0)$, such that $x(0) = x_0$, for sufficiently small δ (depending on x_0).

We now state a theorem which gives the local existence and uniqueness of X^k solutions for $k = 1, 2, \dots$

THEOREM 3.3. *Suppose that $\Psi_c \in C^{k+1}(\mathbf{R})$. We assume that $K_T(\Psi, u)$ is bounded from X^k to \mathbf{R} and X^{k-1} -Lipschitz on X^K -bounded sets, i.e., for all $x_i = (g_i, \Phi_i, \Psi_i) \in X^K$, $i = 1, 2$, $K_T(\Psi, u)$ satisfies*

$$|K_T(\Psi_1, u(x_1)) - K_T(\Psi_2, u(x_2))| \leq C_K \|x_1 - x_2\|_{X^{k-1}},$$

where C_K is a function of X^k norms of x_i , $i = 1, 2$, which is bounded on bounded sets in X^k .

Then the Cauchy problem

$$\frac{dx}{d\xi} = Ax + f(x, u), \quad x(0) = x_0 \in X^k$$

has a unique solution $x \in C(0, \delta; X^k) \cap C^1(0, \delta; X^{k-1})$, such that $x(0) = x_0$, for sufficiently small δ (depending on x_0).

Proof. We only outline the proof.

Since $\Psi_c \in C^{k+1}$ and the underlying space has only one dimension, it follows from the Sobolev embedding theorem that for $\Phi \in \mathbf{R}$ we have that the mapping

$$\Psi_c(\cdot) - \overline{\Psi_c(\cdot)} : \mathcal{H}^k \rightarrow H^k$$

is C^1 for $k > \frac{1}{2}$. (See McOwen, [13, p. 221].) (Here, \mathcal{H}^k denotes the usual Sobolev space on the unit circle.) In particular, this mapping is locally X^{k-1} -Lipschitz X^k -bounded sets and because of

$$\begin{aligned} \|f(x)\|_{X^k}^2 &\leq \|f(0)\|_{X^k}^2 + \|f(x) - f(0)\|_{X^k}^2, \\ &\leq \|f(0)\|_{X^k}^2 + C_L \|x\|_{X^k}^2 \end{aligned}$$

it is also bounded on bounded sets. The result follows. \square

3.3. A priori estimates for X^1 solutions. We have Proposition 3.1.

PROPOSITION 3.1. *Assume that Ψ_c is a C^2 function. Let X^1, X^0, f , and A be as in section 3.2, and let $x = (g, \Phi, \Psi) \in C(0, \delta; X^1) \cap C^1(0, \delta; X^0)$ be a solution to (22) for some $\delta > 0$. Then*

$$(23) \quad \frac{d}{d\xi} \frac{1}{2} \|g\|_{H^1_K}^2 = a \int_0^{2\pi} \Psi'_c(\Phi + g) \left(\frac{\partial g}{\partial \theta} \right)^2 d\theta.$$

Proof. In the proof we will deal with the expression $\frac{\partial^2 g}{\partial \theta^2}$ which is not in L^2 for all X^1 functions. Therefore, we first need to prove that (23) holds on a dense subset of X^1 solutions of (22) for which $\frac{\partial^2 g}{\partial \theta^2}$ makes sense. For this subset we choose X^2 solutions of (22).

Assume that $x = (g, \Phi, \Psi) \in C(0, \delta; X^2) \cap C^1(0, \delta; X^1)$ is a solution to (22) for some $\delta > 0$. Then

$$\frac{d}{d\xi} \frac{1}{2} \|g\|_{H^1_K}^2 = \frac{1}{2} \left(\left\langle \frac{d}{d\xi} \frac{\partial g}{\partial \theta}, K \frac{\partial g}{\partial \theta} \right\rangle + \left\langle \frac{\partial g}{\partial \theta}, \frac{d}{d\xi} K \frac{\partial g}{\partial \theta} \right\rangle \right).$$

Since K is self-adjoint and $\frac{d}{d\xi}$, K , and $\frac{\partial}{\partial \theta}$ commute, we have

$$\frac{d}{d\xi} \frac{1}{2} \|g\|_{H^1_K}^2 = \left\langle \frac{\partial g}{\partial \theta}, \frac{\partial}{\partial \theta} K \frac{d}{d\xi} g \right\rangle.$$

Thus we have by (9),

$$\frac{d}{d\xi} \frac{1}{2} \|g\|_{H^1_K}^2 = a \left\langle \frac{\partial \Psi_c(\Phi + g)}{\partial \theta}, \frac{\partial g}{\partial \theta} \right\rangle - a \left\langle \frac{\partial \bar{\Psi}_c}{\partial \theta}, \frac{\partial g}{\partial \theta} \right\rangle - \frac{1}{2} \left\langle \frac{\partial^2 g}{\partial \theta^2}, \frac{\partial g}{\partial \theta} \right\rangle.$$

One has $\frac{\partial \bar{\Psi}_c}{\partial \theta} = 0$. Moreover,

$$\frac{1}{2} \left\langle \frac{\partial^2 g}{\partial \theta^2}, \frac{\partial g}{\partial \theta} \right\rangle = \int_0^{2\pi} \frac{\partial}{\partial \theta} \left(\frac{\partial g}{\partial \theta} \right)^2 d\theta = 0.$$

Thus,

$$\begin{aligned} \frac{d}{d\xi} \frac{1}{2} \|g\|_{H^1_K}^2 &= a \left\langle \frac{\partial \Psi_c(\Phi + g)}{\partial \theta}, \frac{\partial g}{\partial \theta} \right\rangle \\ &= a \int_0^{2\pi} \Psi'_c(\Phi + g) \left(\frac{\partial g}{\partial \theta} \right)^2 d\theta \end{aligned}$$

for $x \in X^2$. Since X^2 local solutions of (22) are dense in the set of X^1 local solutions of (22), if we can show that the right-hand side of (23) is X^1 continuous, then (23) will hold for all X^1 solutions of (22). Let \mathcal{M} be a bounded subset of X^1 and let $x_1 = (g_1, \Phi_1, \Psi_1) \in \mathcal{M}$ and $x_2 = (g_2, \Phi_2, \Psi_2) \in \mathcal{M}$ be arbitrary. Then, by Lemma 2.1, for every θ , $\Phi_1 + g_1(\theta)$ and $\Phi_2 + g_2(\theta)$ belong to a bounded interval $I_{\mathcal{M}} \subset \mathbf{R}$. Therefore, using Lemma 2.3 one obtains

$$\begin{aligned} &|\Psi'_c(\Phi_1 + g_1) - \Psi'_c(\Phi_2 + g_2)| \\ &\leq \left| \int_0^1 \Psi''_c(\Phi_2 + g_2 + s(\Phi_1 - \Phi_2 + g_1 - g_2)) ds (\Phi_1 - \Phi_2 + g_1 - g_2) \right| \\ &\leq \sup_{\phi \in I_{\mathcal{M}}} |\Psi''_c(\phi)| (|\Phi_1 - \Phi_2| + \|g_1 - g_2\|_{L^\infty}) \\ &\leq \sup_{\phi \in I_{\mathcal{M}}} |\Psi''_c(\phi)| (|\Phi_1 - \Phi_2| + \frac{\sqrt{\pi}}{\sqrt{6}} \|g_1 - g_2\|_{H^1_K}) \\ &\leq \sup_{\phi \in I_{\mathcal{M}}} |\Psi''_c(\phi)| \|x_1 - x_2\|_{X^1}. \end{aligned}$$

We can now use this to show that the right-hand side of (23) is continuous in X^1 :

$$\begin{aligned} & \left| \int_0^{2\pi} \Psi'_c(\Phi_1 + g_1) \left(\frac{\partial g_1}{\partial \theta}\right)^2 d\theta - \int_0^{2\pi} \Psi'_c(\Phi_2 + g_2) \left(\frac{\partial g_2}{\partial \theta}\right)^2 d\theta \right| \\ & \leq \int_0^{2\pi} \left| \Psi'_c(\Phi_1 + g_1) \left(\left(\frac{\partial g_1}{\partial \theta}\right)^2 - \left(\frac{\partial g_2}{\partial \theta}\right)^2 \right) \right| + |\Psi'_c(\Phi_1 + g_1) - \Psi'_c(\Phi_2 + g_2)| \left(\frac{\partial g_2}{\partial \theta}\right)^2 d\theta \\ & \leq \sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)| (\|g_1\|_{H^1}^2 - \|g_2\|_{H^1}^2) + \sup_{\phi \in I_{\mathcal{M}}} |\Psi''_c(\phi)| \|x_1 - x_2\|_{X^1} \|g_2\|_{H^1} \\ & \leq (\sup_{\phi \in I_{\mathcal{M}}} |\Psi'_c(\phi)| (\|g_1\|_{H^1_K} + \|g_2\|_{H^1_K})) + \sup_{\phi \in I_{\mathcal{M}}} |\Psi''_c(\phi)| \|g_2\|_{H^1_K} \|x_1 - x_2\|_{X^1}. \end{aligned}$$

Thus (23) holds for all X^1 solutions of (22). \square

COROLLARY 3.1. H^1_K solutions of (9) grow at most exponentially; i.e., there is no finite-time blow-up of H^1_K solutions of (9). In particular, one has

$$(24) \quad \frac{d}{d\xi} \|g\|_{H^1_K}^2 \leq a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) \|g\|_{H^1_K}^2$$

and

$$(25) \quad \frac{d}{d\xi} \|g\|_{H^1_K} \leq a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) \|g\|_{H^1_K}.$$

Proof. Observe that it follows from our assumptions that Ψ'_c is bounded from above. Now, using Proposition 3.1 and Lemma 2.2 one obtains

$$\frac{d}{d\xi} \|g\|_{H^1_K}^2 \leq a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) \|g\|_{H^1_K}^2.$$

Observe that $\frac{\partial}{\partial \xi} \frac{1}{2} \|g\|_{H^1_K}^2 = \|g\|_{H^1_K} \frac{\partial}{\partial \xi} \|g\|_{H^1_K}$ so upon dividing (24) by $\|g\|_{H^1_K}$ we get (25).

By Grönwall’s lemma we get that the solutions grow at most exponentially. \square

3.4. Global existence and uniqueness of X^1 solutions. In section 4 we will construct a globally stabilizing feedback control u for the system (9), (10), and (11). As a consequence, the global existence of solutions of the system (9), (10), and (11) will be established. The main condition on characteristic Ψ_c is

$$\sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) < +\infty;$$

i.e., the positive slopes of the characteristic are bounded. Note that this condition follows from our assumptions about the characteristic stated in the beginning of section 2.1.

THEOREM 3.4. Assume that X^1, X^0, f , and A are as before. Assume that

$$\sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) < \infty$$

and there exist constants N_1, N_2 such that $|K_T(\Psi, u(x))| \leq N_1 + N_2 \|x\|_{X^1}$. Then for any $T > 0$ the Cauchy problem

$$(26) \quad \frac{dx}{d\xi} = Ax + f(x), \quad x(0) = x_0 \in X^1$$

has a unique solution $x \in C(0, T; X^1) \cap C^1(0, T; X^0)$ such that $x(0) = x_0$.

Proof. Since the derivative of the characteristic is bounded from above, there exist positive constants L_1, L_2 such that $\Psi_c(\phi) > -L_1 - L_2|\phi|$ when $\phi < 0$ and $\Psi_c(\phi) < L_1 + L_2\phi$ for $\phi > 0$. We now get

$$\begin{aligned} \Phi \bar{\Psi}_c &= \frac{1}{2\pi} \int_0^{2\pi} \Phi \Psi_c(\Phi + g) d\theta \\ &\leq |\Phi|(L_1 + L_2(|\Phi| + \|g\|_{L^\infty})) \\ &\leq L_1(1 + |\Phi|^2) + L_2|\Phi|^2 + L_2|\Phi|\|g\|_{L^\infty} \\ &\leq L_1(1 + |\Phi|^2) + L_2|\Phi|^2 + L_2\frac{\sqrt{\pi}}{\sqrt{6}}\frac{1}{2}(\|g\|_{H^1}^2 + |\Phi|^2). \end{aligned}$$

Therefore,

$$(27) \quad \Phi \bar{\Psi}_c \leq L_1 + \left(L_1 + L_2 \left(1 + \frac{\sqrt{\pi}}{\sqrt{6}} \frac{1}{2} \right) \right) \|x\|_{X^1}^2.$$

Now by Corollary 3.1 we have

$$\begin{aligned} \frac{d}{d\xi} \frac{1}{2} \|x\|_{X^1}^2 &\leq a \sup_{\mathbf{R}} \Psi'_c \|g\|_{H^1_K}^2 + \Phi \frac{d}{d\xi} \Phi + \Psi \frac{d}{d\xi} \Psi \\ &= a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) \|g\|_{H^1_K}^2 + \frac{1}{l_c} \Phi \bar{\Psi}_c - \frac{1}{l_c} \Phi \Psi + \frac{1}{4l_c B^2} \Phi \Psi \\ &\quad - \frac{1}{4l_c B^2} \Psi K_T(\Psi, u(x)) \\ &\leq a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) \|g\|_{H^1_K}^2 + L_1 + \left(L_1 + L_2 \left(1 + \frac{\sqrt{\pi}}{\sqrt{6}} \frac{1}{2} \right) \right) \|x\|_{X^1}^2 \\ &\quad + \left(\frac{1}{l_c} + \frac{1}{4l_c B^2} \right) \frac{1}{2} (|\Phi|^2 + |\Psi|^2) + \frac{1}{4l_c B^2} |\Psi K_T(\Psi, u(x))| \\ &\leq a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) \|g\|_{H^1_K}^2 + L_1 + \left(L_1 + L_2 \left(1 + \frac{\sqrt{\pi}}{\sqrt{6}} \frac{1}{2} \right) \right) \|x\|_{X^1}^2 \\ &\quad + \left(\frac{1}{l_c} + \frac{1}{4l_c B^2} \right) \frac{1}{2} (|\Phi|^2 + |\Psi|^2) + \frac{1}{4l_c B^2} (N_1 + (N_1 + N_2) \|x\|_{X^1}^2). \end{aligned}$$

Therefore, we obtain

$$(28) \quad \frac{d}{d\xi} \frac{1}{2} \|x\|_{X^1}^2 \leq C_1 + C_2 \|x\|_{X^1}^2.$$

Here

$$\begin{aligned} C_1 &= \frac{1}{4l_c B^2} N_1 + L_1, \\ C_2 &= a \sup_{\phi \in \mathbf{R}} \Psi'_c(\phi) + L_1 + L_2 \left(1 + \frac{\sqrt{\pi}}{\sqrt{6}} \frac{1}{2} \right) + \left(\frac{1}{l_c} + \frac{1}{4l_c B^2} \right) \frac{1}{2} + \frac{N_1 + N_2}{4l_c B^2}. \end{aligned}$$

By Grönwall’s lemma we now get

$$(29) \quad \|x\|_{X^1}^2(\xi) \leq \left(\|x\|_{X^1}^2(0) + \frac{C_1}{C_2} \right) e^{C_2 \xi} - \frac{C_1}{C_2}.$$

We therefore see that solutions of (22) are bounded for all finite times and thus we have a global solution. \square

Because of the embedding $H^1 \hookrightarrow L^\infty$, this means that L^∞ norms of H^1 solutions do not blow up in finite time either.

Global existence and uniqueness of H^1_K solutions of the full Moore–Greitzer model and the corresponding a priori estimates allow us to construct a controller that globally stabilizes the peak or any axisymmetric equilibrium to the right of the peak. The controller will have a similar form to a controller for an MG3 model with the H^1_K norm of the stall variable replacing the magnitude of the first Fourier mode of the stall cell.

4. H^1 backstepping. We are going to construct a feedback controller stabilizing the peak or any axisymmetric equilibrium to the right of the peak of the characteristic for the full Moore–Greitzer model. The feedback is constructed by the following backstepping procedure. In the first step we define a positive definite function $V_1(g)$ and construct a function $\widehat{\Phi}(\|g\|)$ such that for $\Phi = \widehat{\Phi}(\|g\|) V_1(g)$ is a Lyapunov function for (9). $V_1(g)$ is called a *control Lyapunov function* and $\widehat{\Phi}(\|g\|)$ is called a *virtual control* for (9). In the second step we define a control Lyapunov function $V_2(\Phi, g)$ and a virtual control $\Psi = \widehat{\Psi}(\|g\|, \Phi)$ for (9) and (10). In the third (and last) step we construct the control Lyapunov function for the full system (9), (10), and (11) with the throttle function u being the control variable. We will refer to this procedure as H^1 backstepping. The obtained feedback control law u uses the H_K^1 norm of a stall cell and resembles familiar control laws for MG3 (see [3], [8], [9], [10], [16], [1]) with A_1 replaced with the H_K^1 norm of g . In terms of the Fourier coefficients A_i of g , this norm is $(\sum_{p=1}^\infty (1 + \frac{am}{i})(iA_i)^2)^{\frac{1}{2}}$. To simplify notation, let us from now on denote the norm $\|\cdot\|_{H_K^1}$ by $\|\cdot\|$.

4.1. H^1 backstepping: Step 1. As a control Lyapunov function for (9) we will use the H_K^1 norm of g . Let

$$V_1(g) := \frac{1}{2} \|g\|^2.$$

We will show that $\frac{d}{d\xi} V_1(g)$ can be made negative definite by a virtual control of the form

$$(30) \quad \Phi = \widehat{\Phi}(\|g\|) = \Gamma + \bar{c}_g \|g\|$$

for $\Gamma \geq \Phi_0$ and sufficiently large positive \bar{c}_g . (Φ_0 denotes the value of the mass flow coefficient at the peak.) In this paper we assume that $\bar{c}_g \geq 0$. We will need the following result.

LEMMA 4.1. *For every $\theta \in [0, 2\pi]$,*

$$\left(\bar{c}_g - \frac{\sqrt{\pi}}{\sqrt{6}}\right) \|g\| \leq \bar{c}_g \|g\| + g \leq \left(\bar{c}_g + \frac{\sqrt{\pi}}{\sqrt{6}}\right) \|g\|.$$

Proof. Note that $\bar{c}_g \|g\| + g \geq \bar{c}_g \|g\| - \|g\|_{L^\infty} \geq (\bar{c}_g - \frac{\sqrt{\pi}}{\sqrt{6}}) \|g\|$ by Lemmas 2.1 and 2.2. \square

Let

$$e_\Phi := \Phi - \widehat{\Phi}(\|g\|) = \Phi - \Gamma - \bar{c}_g \|g\|.$$

It follows from Proposition 3.1 and Lemma 2.3 that one can represent $\frac{d}{d\xi} V_1(g)$ as

$$\begin{aligned} \frac{d}{d\xi} V_1(g) &= a \int_0^{2\pi} \Psi'_c(\widehat{\Phi}(\|g\|) + g) \left(\frac{\partial g}{\partial \theta}\right)^2 d\theta \\ &+ a \left(\int_0^{2\pi} \left(\int_0^1 \Psi''_c(\widehat{\Phi}(\|g\|) + se_\Phi + g) ds\right) \left(\frac{\partial g}{\partial \theta}\right)^2 d\theta\right) e_\Phi. \end{aligned}$$

Using Lemma 2.3 again, one obtains

$$\begin{aligned} \frac{d}{d\xi} V_1(g) &= a \int_0^{2\pi} (\Psi'_c(\Gamma) + \int_0^1 \Psi''_c(\Gamma + s(\bar{c}_g \|g\| + g)) ds (\bar{c}_g \|g\| + g)) \left(\frac{\partial g}{\partial \theta}\right)^2 d\theta \\ &+ a \left(\int_0^{2\pi} \left(\int_0^1 \Psi''_c(\widehat{\Phi}(\|g\|) + se_\Phi + g) ds\right) \left(\frac{\partial g}{\partial \theta}\right)^2 d\theta\right) e_\Phi \\ &= a \Psi'_c(\Gamma) \|g\|_{H^1}^2 \\ &+ a \int_0^{2\pi} \left(\int_0^1 \Psi''_c(\Gamma + s(\bar{c}_g \|g\| + g)) ds (\bar{c}_g \|g\| + g)\right) \left(\frac{\partial g}{\partial \theta}\right)^2 d\theta \\ &+ a \left(\int_0^{2\pi} \left(\int_0^1 \Psi''_c(\widehat{\Phi}(\|g\|) + se_\Phi + g) ds\right) \left(\frac{\partial g}{\partial \theta}\right)^2 d\theta\right) e_\Phi. \end{aligned}$$

Note that if $\Gamma > \Phi_{infl}$ then $\int_0^1 \Psi_c''(\Gamma + s(\bar{c}_g \|g\| + g)) ds$ can be bounded from above by a negative constant that depends only on Γ . Namely,

$$(31) \quad \int_0^1 \Psi_c''(\Gamma + s(\bar{c}_g \|g\| + g)) ds \leq \sup_{\Gamma \leq \phi} \Psi_c''(\phi) < 0.$$

Define

$$c_1 := \frac{a}{1 + am} \left(\bar{c}_g - \frac{\sqrt{\pi}}{\sqrt{6}} \right) \sup_{\Gamma \leq \phi} \Psi_c''(\phi),$$

$$c_2(\Phi, \|g\|) := a \sup_{\min(\widehat{\Phi}(\|g\|), \Phi) - \|g\|_{L^\infty} \leq \phi \leq \max(\widehat{\Phi}(\|g\|), \Phi) + \|g\|_{L^\infty}} |\Psi_c''(\phi)|.$$

PROPOSITION 4.1. *Assume that $\Gamma \geq \Phi_0$ and $\bar{c}_g > \frac{\sqrt{\pi}}{\sqrt{6}}$. Then $\Psi_c'(\Gamma) \leq 0$, $c_1 < 0$,*

$$(32) \quad \begin{aligned} \frac{d}{d\xi} V_1(g) &\leq \left(\frac{a}{1+am} \Psi_c'(\Gamma) + c_1 \|g\| \right) \|g\|^2 \\ &+ c_2(\Phi, \|g\|) \|g\|^2 |e_\Phi|, \end{aligned}$$

and

$$(33) \quad \begin{aligned} \frac{d}{d\xi} \|g\| &\leq \left(\frac{a}{1+am} \Psi_c'(\Gamma) + c_1 \|g\| \right) \|g\| \\ &+ c_2(\Phi, \|g\|) \|g\| |e_\Phi|. \end{aligned}$$

Proof. Note that it follows from Lemma 4.1 that $\bar{c}_g \|g\| + g \geq (\bar{c}_g - \frac{\sqrt{\pi}}{\sqrt{6}}) \|g\|$. Moreover, since $\Gamma > \Phi_{infl}$, (31) holds. Therefore,

$$\begin{aligned} \frac{d}{d\xi} V_1(g) &\leq (a \Psi_c'(\Gamma) + a(\bar{c}_g - \frac{\sqrt{\pi}}{\sqrt{6}}) \sup_{\Gamma \leq \phi} \Psi_c''(\phi) \|g\|) \|g\|_{H^1}^2 \\ &+ c_2(\Phi, \|g\|) \|g\|_{H^1}^2 |e_\Phi|. \end{aligned}$$

The first term is nonpositive; the last one is positive. Therefore, the inequality (32) follows from Lemma 2.2. Now the inequality (33) follows from (32). \square

Proposition 4.1 is the most important result. It allows us to carry out the backstepping procedure for an infinite-dimensional system (9), (10), and (11) without the necessity of working with its infinite-dimensional part (9). What we have done here is a *replacement of an infinite-dimensional evolution equation (9) with two finite-dimensional differential inequalities (32) and (33)*. As we shall see this replacement makes the next two backsteps quite standard.

In particular, for $\Phi = \widehat{\Phi}(\|g\|)$ one obtains

$$\frac{d}{d\xi} V_1(g) \leq \left(\frac{a}{1 + am} \Psi_c'(\Gamma) + c_1 \|g\| \right) \|g\|^2.$$

If $\Gamma \geq \Phi_0$ then $\Psi_c'(\Gamma) \leq 0$, $c_1 < 0$, and hence for $\Phi = \widehat{\Phi}(\|g\|)$ $\frac{d}{d\xi} V_1(g)$ is negative definite. For $\|g\|$ small, if $\Gamma > \Phi_0$ then $\Psi_c'(\Gamma) < 0$ and $\frac{d}{d\xi} V_1(g)$ depends quadratically on $\|g\|$, whereas for $\Gamma = \Phi_0$ one has $\Psi_c'(\Gamma) = 0$ and therefore the dependence of $\frac{d}{d\xi} V_1(g)$ on $\|g\|$ is cubic.

Remark 4.1. Note that the essential property of the virtual control (30) that allows us to make the time derivative of the control Lyapunov function negative

was the ability of moving the stall cell “over the top,” so that the whole mass flow $\widehat{\Phi}(\|g\|) + g$ is to the right of the peak, where the slope of the characteristic is negative. The Sobolev embedding was used to guarantee that property. A natural question is why did we not use the L^∞ norm of the stall cell or its minimum value instead of the H_K^1 norm. The reason is that in the next step of the backstepping procedure we will need a bound on the time derivative of whatever norm of the stall cell we use in the first step. We have such information about the time derivative of the H_K^1 norm, but we do not yet have the information about the L^∞ norm of the stall cell. We are currently working on the design that uses L^∞ norm of the stall cell or its minimum in the first step of backstepping.

4.2. H_1 backstepping: Step 2. As a control Lyapunov function for (9) and (10) we will use

$$V_2(\Phi, g) := \frac{1}{2}\|g\|^2 + \frac{1}{2}e_\Phi^2.$$

We will show that $\frac{d}{d\xi} V_2(\Phi, g)$ can be made negative definite by a virtual control of the form

$$\begin{aligned} \Psi &= \widehat{\Psi}(\|g\|, \Phi) \\ &= \Psi_c(\Gamma) + \bar{c}_\Phi(\|g\|, \Phi)e_\Phi \end{aligned}$$

for sufficiently large $\bar{c}_\Phi(\|g\|, \Phi)$. (For semiglobal stabilization $\bar{c}_\Phi(\|g\|, \Phi)$ can be chosen to be a constant depending on the desired region of operation.)

Let

$$\begin{aligned} e_\Psi &:= \Psi - \widehat{\Psi}(\|g\|, \Phi) \\ &= \Psi - \Psi_c(\Gamma) - \bar{c}_\Phi(\|g\|, \Phi)e_\Phi. \end{aligned}$$

To calculate $\frac{d}{d\xi} V_2(\Phi, g)$ we will need to express $\frac{d}{d\xi} e_\Phi$ in terms of e_Φ , e_Ψ , and g . For this, note that

$$\begin{aligned} \frac{d}{d\xi} e_\Phi &= \frac{d}{d\xi} \Phi - \frac{d}{d\xi} \widehat{\Phi}(\|g\|) \\ &= \frac{1}{l_c}(\overline{\Psi_c(\Phi + g)} - \Psi) - \bar{c}_g \frac{d}{d\xi} \|g\|. \end{aligned}$$

Applying Lemma 2.3 twice, one obtains

$$\begin{aligned} \frac{d}{d\xi} e_\Phi &= \frac{1}{l_c}(\Psi_c(\Gamma) + \int_0^{2\pi} (\int_0^1 \Psi'_c(\Gamma + s(\bar{c}_g\|g\| + g))ds)(\bar{c}_g\|g\| + g)d\theta \\ &\quad + (\int_0^{2\pi} (\int_0^1 \Psi'_c(\widehat{\Phi}(\|g\|) + se_\Phi + g)ds)d\theta)e_\Phi \\ &\quad - \widehat{\Psi}(\|g\|, \Phi) - e_\Psi - \bar{c}_g \frac{d}{d\xi} \|g\| \\ &= \frac{1}{l_c}(\int_0^{2\pi} (\int_0^1 \Psi'_c(\Gamma + s(\bar{c}_g\|g\| + g))ds)(\bar{c}_g\|g\| + g)d\theta \\ &\quad + (\int_0^{2\pi} (\int_0^1 \Psi'_c(\widehat{\Phi}(\|g\|) + se_\Phi + g)ds)d\theta)e_\Phi \\ &\quad - \bar{c}_\Phi(\|g\|, \Phi)e_\Phi - e_\Psi - \bar{c}_g \frac{d}{d\xi} \|g\|. \end{aligned}$$

To simplify calculations, we introduce the following notation:

$$\begin{aligned} c_0 &:= \frac{a}{1+am} \Psi'_c(\Gamma), \\ c_3(g) &:= |(\bar{c}_g + 1) \int_0^{2\pi} \int_0^1 \Psi'_c(\Gamma + s(\bar{c}_g\|g\| + g))dsd\theta|, \\ c_4(\Phi, g) &:= \int_0^{2\pi} \int_0^1 \Psi'_c(\widehat{\Phi}(\|g\|) + se_\Phi + g)dsd\theta. \end{aligned}$$

Note that $c_3(g)$ and $c_4(\Phi, g)$ can be bounded by functions of $\|g\|$. For clarity, in the following calculations we use notation c_i instead of $c_i(g)$, etc.

LEMMA 4.2. *Assume that $\Gamma \geq \Phi_0$ and $\bar{c}_g > \frac{\sqrt{\pi}}{\sqrt{6}}$. Then*

$$\begin{aligned} \left| \frac{d}{d\xi} e_\Phi \right| &\leq \frac{1}{l_c} (c_3 \|g\| + (c_4 - \bar{c}_\Phi) |e_\Phi| + |e_\Psi|) \\ &\quad + \bar{c}_g (|(c_0 + c_1 \|g\|)| \|g\| + c_2 \|g\| |e_\Phi|). \end{aligned}$$

and

$$\begin{aligned} e_\Phi \frac{d}{d\xi} e_\Phi &\leq \frac{1}{l_c} (c_3 \|g\| |e_\Phi| + (c_4 - \bar{c}_\Phi) e_\Phi^2 + |e_\Phi e_\Psi|) \\ &\quad + \bar{c}_g (|(c_0 + c_1 \|g\|)| \|g\| |e_\Phi| + c_2 \|g\| e_\Phi^2). \end{aligned}$$

Therefore, assuming $\Gamma \geq \Phi_0$ and $\bar{c}_g > \frac{\sqrt{\pi}}{\sqrt{6}}$ one has

$$\begin{aligned} \frac{d}{d\xi} V_2(\Phi, g) &= \frac{d}{d\xi} V_1(g) + e_\Phi \frac{d}{d\xi} e_\Phi \\ &\leq (c_0 + c_1 \|g\|) \|g\|^2 + c_2 \|g\|^2 |e_\Phi| \\ &\quad + \frac{1}{l_c} (c_3 \|g\| |e_\Phi| + (c_4 - \bar{c}_\Phi) e_\Phi^2 + |e_\Phi e_\Psi|) \\ &\quad + \bar{c}_g (|(c_0 + c_1 \|g\|)| \|g\| |e_\Phi| + c_2 \|g\| e_\Phi^2) \\ &\leq (c_0 + c_1 \|g\|) \|g\|^2 \\ &\quad + (c_2 \|g\| + \frac{1}{l_c} c_3 + \bar{c}_g |(c_0 + c_1 \|g\|)|) \|g\| |e_\Phi| \\ &\quad + (\frac{1}{l_c} (c_4 - \bar{c}_\Phi) + \bar{c}_g c_2 \|g\|) e_\Phi^2 + \frac{1}{l_c} |e_\Phi e_\Psi|. \end{aligned}$$

Define

$$c_5 := (\bar{c}_g + 1) |\Psi'_c(\Gamma)|,$$

$$c_6(g) := (\bar{c}_g + 1) \int_0^{2\pi} \int_0^1 \int_0^1 \Psi''_c(\Gamma + s_1 s_2 (\bar{c}_g \|g\| + g)) ds_1 ds_2 d\theta.$$

Note that $c_6(g)$ can be bounded by functions of $\|g\|$. It follows from Lemma 2.3 that

$$c_3 \leq c_5 + c_6 \|g\|.$$

Therefore, one obtains

$$\begin{aligned} \frac{d}{d\xi} V_2(\Phi, g) &\leq (c_0 + c_1 \|g\|) \|g\|^2 \\ &\quad + (c_2 \|g\| + \frac{1}{l_c} (c_5 + c_6 \|g\|) + \bar{c}_g |(c_0 + c_1 \|g\|)|) \|g\| |e_\Phi| \\ &\quad + (\frac{1}{l_c} c_4 + \bar{c}_g c_2 \|g\| - \frac{1}{l_c} \bar{c}_\Phi) e_\Phi^2 + \frac{1}{l_c} |e_\Phi e_\Psi|. \end{aligned}$$

Hence, we have the following result.

PROPOSITION 4.2. *Assume that $\Gamma \geq \Phi_0$ and $\bar{c}_g > \frac{\sqrt{\pi}}{\sqrt{6}}$. Then*

$$\begin{aligned} \frac{d}{d\xi} V_2(\Phi, g) &\leq (c_0 + c_1 \|g\|) \|g\|^2 \\ &\quad + (\bar{c}_g |c_0| + \frac{1}{l_c} c_5 + (c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|) \|g\|) \|g\| |e_\Phi| \\ &\quad + (\frac{1}{l_c} (c_4 - \bar{c}_\Phi) + \bar{c}_g c_2 \|g\|) e_\Phi^2 + \frac{1}{l_c} |e_\Phi e_\Psi|. \end{aligned}$$

Assume that $\Gamma \geq \Phi_0$ and $\bar{c}_g > \frac{\sqrt{\pi}}{\sqrt{6}}$. Observe that if $\Psi = \widehat{\Psi}(\|g\|, \Phi)$ then

$$\begin{aligned} \frac{d}{d\xi} V_2(\Phi, g) &\leq (c_0 + c_1 \|g\|) \|g\|^2 \\ &\quad + (\bar{c}_g |c_0| + \frac{1}{l_c} c_5 + (c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|) \|g\|) \|g\| |e_\Phi| \\ &\quad + (\frac{1}{l_c} (c_4 - \bar{c}_\Phi) + \bar{c}_g c_2 \|g\|) e_\Phi^2 \\ &= c_{11} \|g\|^2 + 2c_{12} \|g\| |e_\Phi| + c_{22} e_\Phi^2, \end{aligned} \tag{34}$$

where

$$\begin{aligned} c_{11} &:= (c_0 + c_1 \|g\|), \\ c_{12} &:= \frac{1}{2}(\bar{c}_g |c_0| + \frac{1}{l_c} c_5 + (c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|) \|g\|), \\ c_{22} &:= (\frac{1}{l_c} (c_4 - \bar{c}_\Phi) + \bar{c}_g c_2 \|g\|). \end{aligned}$$

Note that the right-hand side of (34) is a quadratic form in $\|g\|$ and e_Φ (with coefficients being functions of $\|g\|$ and e_Φ). This quadratic form can be made negative definite by choosing sufficiently large \bar{c}_Φ . A sufficient condition for $\frac{d}{d\xi} V_2(\Phi, g)$ to be negative definite for $\Psi = \widehat{\Psi}(\|g\|, \Phi)$ is

$$\begin{aligned} \Delta_1 &:= c_{11} < 0, \\ \Delta_2 &:= c_{11} c_{22} - c_{12}^2 > 0, \end{aligned}$$

which is satisfied if

$$(35) \quad \bar{c}_\Phi \geq c_4 + l_c \bar{c}_g c_2 \|g\| + l_c \frac{(\bar{c}_g |c_0| + \frac{1}{l_c} c_5 + (c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|) \|g\|)^2}{4|c_0 + c_1 \|g\|}.$$

Observe that $\Gamma = \Phi_0$ implies that $c_0 = 0$, so that it may seem that the gain function \bar{c}_Φ blows up when $g = 0$. However, this is not the case. Note that $\Gamma = \Phi_0$ also implies that $\bar{c}_g |c_0| + \frac{1}{l_c} c_5 = 0$. Therefore, for $\Gamma = \Phi_0$ the right-hand side of the inequality (35) becomes

$$c_4 + l_c \bar{c}_g c_2 \|g\| + l_c \frac{(c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|)^2 \|g\|^2}{4|c_1| \|g\|}.$$

For $\|g\| = 0$ this quantity is not defined. However, it has a finite limit c_4 at $\|g\| = 0$. Hence, one can conclude that $V_2(\Phi, g)$ is a valid control Lyapunov function also for $\Gamma = \Phi_0$.

4.3. H^1 backstepping: Step 3. As a control Lyapunov function for full model (9), (10), and (11) we will use

$$V_3(\Phi, \Psi, g) := \frac{1}{2} \|g\|^2 + \frac{1}{2} e_\Phi^2 + \frac{4l_c B^2}{2} e_\Psi^2.$$

We will show that $\frac{d}{d\xi} V_3(\Phi, \Psi, g)$ can be made negative definite by a throttle control of the form

$$(36) \quad K_T(\Psi, u) = \Phi + \bar{c}_\Psi(\|g\|, \Phi) e_\Psi$$

for sufficiently large $\bar{c}_\Psi(\|g\|, \Phi)$. (For semiglobal stabilization $\bar{c}_\Psi(\|g\|, \Phi)$ can be chosen to be a constant depending on the desired region of operation.)

To calculate $\frac{d}{d\xi} V_3(\Phi, \Psi, g)$ we will need to express $\frac{d}{d\xi} e_\Psi$ in terms of e_Φ , e_Ψ , and g . (Throughout the paper, for simplicity, we skip the arguments of functions.) One has

$$\begin{aligned} \frac{d}{d\xi} e_\Psi &= \frac{d}{d\xi} \Psi - \frac{d}{d\xi} \widehat{\Psi} \\ &= \frac{1}{4l_c B^2} (-\bar{c}_\Psi e_\Psi) - e_\Phi \frac{d}{d\xi} \bar{c}_\Phi - \bar{c}_\Phi \frac{d}{d\xi} e_\Phi \\ &= \frac{1}{4l_c B^2} (-\bar{c}_\Psi e_\Psi) - \frac{\partial \bar{c}_\Phi}{\partial \|g\|} \frac{d}{d\xi} \|g\| - \frac{\partial \bar{c}_\Phi}{\partial \Phi} \frac{d}{d\xi} \Phi - \bar{c}_\Phi \frac{d}{d\xi} e_\Phi. \end{aligned}$$

Thus, using Proposition 4.1 and Lemma 4.2 one obtains

$$\begin{aligned}
 4l_c B^2 e_\Psi \frac{d}{d\xi} e_\Psi &\leq -\bar{c}_\Psi e_\Psi^2 \\
 &+ 4l_c B^2 (|\frac{\partial \bar{c}_\Phi}{\partial \|g\|}|((c_0 + c_1 \|g\|)) \|g\| |e_\Psi| + c_2 \|g\| |e_\Phi e_\Psi|) \\
 (37) \quad &+ |\frac{\partial \bar{c}_\Phi}{\partial \Phi}| (\frac{1}{l_c} (c_3 \|g\| |e_\Psi| + |(c_4 - \bar{c}_\Phi)| |e_\Phi e_\Psi| + e_\Psi^2)) \\
 &+ \bar{c}_\Phi (\frac{1}{l_c} (c_3 \|g\| |e_\Psi| + |(c_4 - \bar{c}_\Phi)| |e_\Phi e_\Psi| + e_\Psi^2)) \\
 &+ \bar{c}_g (|(c_0 + c_1 \|g\|)| \|g\| |e_\Psi| + c_2 \|g\| |e_\Phi e_\Psi|).
 \end{aligned}$$

Hence, using Proposition 4.2 and (37) one obtains

$$\begin{aligned}
 \frac{d}{d\xi} V_3(\Phi, \Psi, g) &\leq (c_0 + c_1 \|g\|) \|g\|^2 \\
 &+ (\bar{c}_g |c_0| + \frac{1}{l_c} c_5 + (c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|) \|g\|) \|g\| |e_\Phi| \\
 &+ (\frac{1}{l_c} (c_4 - \bar{c}_\Phi) + \bar{c}_g c_2 \|g\|) e_\Phi^2 + \frac{1}{l_c} |e_\Phi e_\Psi| \\
 &- \bar{c}_\Psi e_\Psi^2 \\
 &+ 4l_c B^2 (|\frac{\partial \bar{c}_\Phi}{\partial \|g\|}|((c_0 + c_1 \|g\|)) \|g\| |e_\Psi| + c_2 \|g\| |e_\Phi e_\Psi|) \\
 &+ |\frac{\partial \bar{c}_\Phi}{\partial \Phi}| (\frac{1}{l_c} (c_3 \|g\| |e_\Psi| + |(c_4 - \bar{c}_\Phi)| |e_\Phi e_\Psi| + e_\Psi^2)) \\
 &+ \bar{c}_\Phi (\frac{1}{l_c} (c_3 \|g\| |e_\Psi| + |(c_4 - \bar{c}_\Phi)| |e_\Phi e_\Psi| + e_\Psi^2)) \\
 &+ \bar{c}_g (|(c_0 + c_1 \|g\|)| \|g\| |e_\Psi| + c_2 \|g\| |e_\Phi e_\Psi|) \\
 &= c_{11} \|g\|^2 + 2c_{12} \|g\| |e_\Phi| + 2c_{13} \|g\| |e_\Psi| + c_{22} e_\Phi^2 + 2c_{23} |e_\Phi| |e_\Psi| + c_{33} e_\Psi^2, \\
 (38)
 \end{aligned}$$

where

$$\begin{aligned}
 c_{11} &= (c_0 + c_1 \|g\|), \\
 c_{12} &= \frac{1}{2} (\bar{c}_g |c_0| + \frac{1}{l_c} c_5 + (c_2 + \frac{1}{l_c} c_6 + \bar{c}_g |c_1|) \|g\|), \\
 c_{22} &= (\frac{1}{l_c} (c_4 - \bar{c}_\Phi) + \bar{c}_g c_2 \|g\|), \\
 c_{13} &:= \frac{1}{2} (4l_c B^2 |\frac{\partial \bar{c}_\Phi}{\partial \|g\|}|((c_0 + c_1 \|g\|)) \\
 &\quad + 4B^2 |\frac{\partial \bar{c}_\Phi}{\partial \Phi}| c_3 + 4B^2 \bar{c}_\Phi c_3 + 4l_c B^2 \bar{c}_\Phi \bar{c}_g (|(c_0 + c_1 \|g\|)|), \\
 c_{23} &:= \frac{1}{2} (\frac{1}{l_c} + 4l_c B^2 c_2 \|g\| + 4B^2 |\frac{\partial \bar{c}_\Phi}{\partial \Phi}| |(c_4 - \bar{c}_\Phi)| \\
 &\quad + 4B^2 \bar{c}_\Phi |c_4 - \bar{c}_\Phi| + 4l_c B^2 \bar{c}_\Phi \bar{c}_g c_2 \|g\|), \\
 c_{33} &:= -\bar{c}_\Psi + 4B^2 |\frac{\partial \bar{c}_\Phi}{\partial \Phi}| + 4B^2 \bar{c}_\Phi.
 \end{aligned}$$

Note that the right-hand side of (38) is a quadratic form in $\|g\|$, $|e_\Phi|$, and $|e_\Psi|$ (with coefficients being functions of g and e_Φ that can be bounded by functions of $\|g\|$ and e_Φ). Assuming that $\bar{c}_g > \frac{\sqrt{\pi}}{\sqrt{6}}$ and \bar{c}_Φ satisfies (35), we can make this quadratic form negative definite by choosing sufficiently large \bar{c}_Ψ . Sufficient conditions for $\frac{d}{d\xi} V_3(\Phi, \Psi, g)$ to be negative definite everywhere are

$$\begin{aligned}
 \Delta_1 &= c_{11} < 0, \\
 \Delta_2 &= c_{11} c_{22} - c_{12}^2 > 0, \\
 \Delta_3 &:= c_{33} \Delta_2 + 2c_{12} c_{13} c_{23} - c_{22} c_{13}^2 - c_{11} c_{23}^2 < 0.
 \end{aligned}$$

The condition $\Delta_1 < 0$ is obviously satisfied (see Step 1). To enforce the condition $\Delta_2 > 0$ one should choose \bar{c}_Φ that satisfies (35) (see Step 2). Finally, once \bar{c}_Φ satisfies

the inequality (35), to assure that $\Delta_3 < 0$, at each point, the gain \bar{c}_Ψ should satisfy the inequality

$$(39) \quad \bar{c}_\Psi > 4B^2 \left| \frac{\partial \bar{c}_\Phi}{\partial \Phi} \right| + 4B^2 \bar{c}_\Phi + \frac{2c_{12}c_{13}c_{23} - c_{22}c_{13}^2 - c_{11}c_{23}^2}{\Delta_2}.$$

If $\Gamma > \Phi_0$ then $\Delta_2 > 0$ holds everywhere and hence the right-hand side of the inequality (39) is defined for everywhere (see Step 2).

However, $\Gamma = \Phi_0$ implies that $c_0 = 0$, and thus Δ_2 vanishes if $g = 0$. Therefore, it may seem that the gain function \bar{c}_Ψ blows up when $g = 0$. However, this is not the case. One can show that the quantity $2c_{12}c_{13}c_{23} - c_{22}c_{13}^2 - c_{11}c_{23}^2$ also vanishes if $g = 0$ and $\frac{2c_{12}c_{13}c_{23} - c_{22}c_{13}^2 - c_{11}c_{23}^2}{\Delta_2}$ has a finite limit as $\|g\|$ goes to zero. Hence, one can conclude that $V_3(\Phi, \Psi, g)$ is a valid control Lyapunov function also for $\Gamma = \Phi_0$. (See similar remarks at the end of section 4.2.)

4.4. The case $\Gamma < \Phi_0$. If the position of the peak is unknown or if the characteristic shifts from its nominal position (because of disturbance, etc.), it may happen that $\Gamma < \Phi_0$. In that case it follows from Proposition 3.1 that $(\Phi, \Psi, g) = (\Gamma, \Psi_c(\Gamma), 0)$ is an unstable equilibrium that cannot be stabilized by the virtual control (30). However, one can prove that the controller of the form (36) will guarantee that the dynamics of the closed-loop system are confined to a ball containing $(\Gamma, \Psi_c(\Gamma), 0)$. The radius of the ball can be made arbitrarily small if one can use arbitrarily high gains in the controller. This modification of the gains in the controller in comparison with the case $\Gamma \geq \Phi_0$ is to be expected, as the controller gains proposed for the case $\Gamma \geq \Phi_0$ were not designed to work also in the case $\Gamma < \Phi_0$.

What we present below is a simple, but not necessarily optimal, way of constructing a controller that confines the dynamics to a ball. Our goal was to provide a simple proof that this is possible, not to actually design a controller that is optimal in any sense.

Assume that $\Phi_{infl} < \Gamma < \Phi_0$. We are going to use notation of the previous sections. We need to introduce two new symbols:

$$\begin{aligned} \bar{c}_0 &:= a\Psi'_c(\Gamma), \\ \bar{c}_{11} &:= \bar{c}_0 + c_1\|g\|. \end{aligned}$$

These quantities will replace c_0 and c_{11} , respectively. One can show that

$$\frac{d}{d\xi} V_3(\Phi, \Psi, g) \leq \bar{c}_{11}\|g\|^2 + 2c_{12}\|g\|\|e_\Phi\| + 2c_{13}\|g\|\|e_\Psi\| + c_{22}e_\Phi^2 + 2c_{23}\|e_\Phi\|\|e_\Psi\| + c_{33}e_\Psi^2.$$

Observe that we had to replace c_0 and c_{11} with \bar{c}_0 and \bar{c}_{11} since now $\Psi'_c(\Gamma) > 0$.

It will be also useful to introduce the following notation. Let

$$\overline{DV}_2(\Phi, g) := \bar{c}_{11}\|g\|^2 + 2c_{12}\|g\|\|e_\Phi\| + c_{22}e_\Phi^2.$$

Then

$$\frac{d}{d\xi} V_3(\Phi, \Psi, g) \leq \overline{DV}_2(\Phi, g) + 2c_{13}\|g\|\|e_\Psi\| + 2c_{23}\|e_\Phi\|\|e_\Psi\| + c_{33}e_\Psi^2.$$

Note that $\bar{c}_0 > 0$ and $c_1 < 0$. Therefore, the upper bound on $\frac{d}{d\xi} V_3(\Phi, \Psi, g)$ cannot be made negative everywhere; as for $e_\Phi = e_\Psi = 0$ and $0 < \|g\| < \frac{-\bar{c}_0}{c_1}$ one has

$\bar{c}_{11} > 0$. However, we will show that one can arbitrarily reduce the size of the set where $\frac{d}{d\xi} V_3(\Phi, \Psi, g) > 0$ by using high gains in the controller. This can be accomplished as follows. First, one can arbitrarily reduce the interval on which $\bar{c}_{11} > 0$ by using a high gain \bar{c}_g , which makes c_1 big negative. Second, one can use high gains \bar{c}_Φ and \bar{c}_Ψ , which make c_{22} and c_{33} big negative.

Let ϵ be an arbitrary positive number. We are going to show that by using sufficiently high gains \bar{c}_g , \bar{c}_Φ , and \bar{c}_Ψ one can guarantee that $\frac{d}{d\xi} V_3(\Phi, \Psi, g) < 0$ outside the set $\mathcal{M}_\epsilon := \{(\Phi, \Psi, g) : \|g\| < \epsilon, |e_\Phi| < \epsilon, |e_\Psi| < \epsilon\}$.

Step 1. Choose \bar{c}_g such that for $\|g\| \geq \epsilon$ one has $\bar{c}_{11} \leq -3$.

Step 2. Choose \bar{c}_Φ such that the following conditions (2a) and (2b) are satisfied:

(2a) For $\|g\| \geq \epsilon$ one has $c_{22} \leq \frac{c_{12}^2}{\bar{c}_{11} + 2}$.

Note that for a fixed $\|g\|$, $\overline{DV}_2(\Phi, g)$ can be viewed as a quadratic function of $|e_\Phi|$. One can show using some elementary algebra that our choice of \bar{c}_Φ guarantees that for $\|g\| \geq \epsilon$ one has $\overline{DV}_2(\Phi, g) \leq -2\epsilon^2$.

(2b) For $\|g\| < \epsilon$ one has $\frac{-2c_{12}}{c_{22}} + \sqrt{|\frac{c_{11}}{c_{22}}|} + \sqrt{|\frac{2}{c_{22}}|} < 1$.

One can show using some elementary algebra that this choice guarantees that for $\|g\| < \epsilon$ and $|e_\Phi| \geq \epsilon$ one has $\overline{DV}_2(\Phi, g) \leq -2\epsilon^2$.

Step 3. Choose \bar{c}_Ψ so that the following conditions (3a) and (3b) are satisfied:

(3a) For $\|g\| \geq \epsilon$ or $|e_\Phi| \geq \epsilon$ one has $c_{33} \leq \frac{(c_{13}\|g\| + c_{23}|e_\Phi|)^2}{\overline{DV}_2(\Phi, g) + \epsilon^2}$.

Note that c_{33} is bounded, as the choice of \bar{c}_Φ in Step 2 guarantees that for $\|g\| \geq \epsilon$ or $|e_\Phi| \geq \epsilon$ one has $\overline{DV}_2(\Phi, g) + \epsilon^2 \leq -\epsilon^2$.

One can show that this choice of \bar{c}_Ψ guarantees that for $\|g\| \geq \epsilon$ or $|e_\Phi| \geq \epsilon$ one has $\frac{d}{d\xi} V_3(\Phi, \Psi, g) \leq -\epsilon^2$.

(3b) For $\|g\| < \epsilon$ and $|e_\Phi| < \epsilon$ one has $\frac{-2c_{13} - 2c_{23}}{c_{33}} + \sqrt{\frac{|c_{11}| + |c_{22}| + 2c_{12} + 1}{|c_{33}|}} < 1$.

One can show that this choice guarantees that for $\|g\| < \epsilon$ and $|e_\Phi| < \epsilon$, but for $|e_\Psi| \geq \epsilon$ one has $\frac{d}{d\xi} V_3(\Phi, \Psi, g) \leq -\epsilon^2$.

Therefore, we have the following result.

PROPOSITION 4.3. *Let the gains \bar{c}_g , \bar{c}_Φ , and \bar{c}_Ψ satisfy the conditions stated in Steps 1–3 above. Then outside the set $\mathcal{M}_\epsilon = \{(\Phi, \Psi, g) : \|g\| < \epsilon, |e_\Phi| < \epsilon, |e_\Psi| < \epsilon\}$ one has*

$$\frac{d}{d\xi} V_3(\Phi, \Psi, g) < -\epsilon^2.$$

Therefore, the state of the closed-loop system enters in a finite time the set

$$\mathcal{N}_\epsilon := \left\{ (\Phi, \Psi, g) : V_3(\Phi, \Psi, g) < \frac{4l_c B^2}{2(1 + 4l_c B^2)} \epsilon^2 \right\}.$$

Note that the high gains of the controller presented in this section are required if one wants to reduce the size of the dynamics and in particular of the stall cell, not to stabilize a small stall cell. If one just wants to confine the dynamics to a ball, high gains are not required.

It is not clear at the moment what are the dynamics of the closed-loop system inside the absorbing set \mathcal{N}_ϵ . This issue is currently under investigation.

5. Controllers for Galerkin projections of the full model. In section 4 we constructed a feedback controller stabilizing a peak or any axisymmetric equilibrium

to the right of the peak for the full Moore–Greitzer PDE model. The feedback law is given by (36) and has a general form

$$K_T(\Psi, u) = K_T(\|g\|_{H_K^1}, \Phi, \Psi).$$

In terms of the magnitudes A_p of the Fourier modes of a stall cell g the control law looks like familiar backstepping control laws for MG3 with A_1 replaced with $(\sum_{p=1}^{\infty} (1 + \frac{am}{p})(pA_p)^2)^{\frac{1}{2}}$. An implementation of this control law would require access to an infinite number of modes of a stall cell g , which is practically impossible.

Remark 5.1. The following was communicated to the authors by Richard Murray from Caltech.

The number of accessible modes depends on the number of pressure sensors used to detect a nonaxisymmetric pressure distribution and their distance from the compressor face. Since it requires $2n+1$ sensors to instantaneously detect the first n modes (by fitting a linear combination of spatial sinusoids), the number of sensors gets somewhat large for higher modes. In addition, with the minimal number of sensors, the last mode is pretty noisy.

The Caltech compressor rig has six sensors so that a measurement of up to second mode magnitude is possible. The rig has enough ports to use 16 sensors, which would make it possible to measure the magnitudes of up to the seventh mode.

Another factor is the distance back from the compressor face. Recall that the magnitudes of the Fourier modes of a stall cell fall off by $e^{-\eta n}$ where η is the nondimensional distance from the compressor face. For the Caltech rig, η is about 0.5, so beyond the third or fourth mode one would not be able to pick out the signal from the noise.

Remark 5.2. An alternative to an instantaneous detection of the first n modes by using $2n+1$ sensors would be using fewer sensors and an observer to reconstruct the modes. If the speed of rotation of the stall cell is known, then one can easily verify that the first few modes are observable (even from a single sensor). The fast decay of the higher modes because of the distance of the sensors from the compressor face makes the observability of these modes poor and therefore is still a limiting factor in the number of detectable modes.

If n modes are accessible, a practical implementation of the controller could use $(\sum_{p=1}^n (1 + \frac{am}{p})(pA_p)^2)^{\frac{1}{2}}$, i.e., a truncation of the infinite series $(\sum_{p=1}^{\infty} (1 + \frac{am}{p})(pA_p)^2)^{\frac{1}{2}}$ representing the H_K^1 norm of the stall cell g . A natural question arises: what can we say about the controller of the form (36) that uses first n modes of the stall cell g ? In this section we will show that this controller *stabilizes the Galerkin projection of the full Moore–Greitzer PDE model onto its first n modes*.

It is easy to show that the ODEs describing evolution of the p th mode of g are

$$\begin{aligned} \dot{a}_p &= \frac{p}{p+am} \left(\frac{a}{\pi} \int_0^{2\pi} \Psi_c(\Phi + g) \sin(p\theta) d\theta + \frac{1}{2} p b_p \right), \\ \dot{b}_p &= \frac{p}{p+am} \left(\frac{a}{\pi} \int_0^{2\pi} \Psi_c(\Phi + g) \cos(p\theta) d\theta - \frac{1}{2} p a_p \right) \end{aligned}$$

or, equivalently,

$$\dot{A}_p = \frac{p}{p+am} \left(\frac{a}{\pi} \int_0^{2\pi} \Psi_c(\Phi + g) \sin(p\theta + r_p) d\theta \right),$$

$$\dot{r}_p = \frac{p}{p + am} \left(-\frac{1}{2}p + \frac{a}{\pi A_p} \int_0^{2\pi} \Psi_c(\Phi + g) \cos(p\theta + r_p) d\theta \right),$$

where g is represented as

$$g = \sum_{p=1}^{\infty} (a_p \sin(p\theta) + b_p \cos(p\theta)) = \sum_{p=1}^{\infty} A_p \sin(p\theta + r_p).$$

A Galerkin projection of the stall PDE with n modes would be the set of $2n$ ODEs as above with g replaced with

$$g_n := \sum_{p=1}^n A_p \sin(p\theta + r_p).$$

Hence, an approximate model would consist of equations

$$(40) \quad \dot{A}_p = \frac{p}{p + am} \left(\frac{a}{\pi} \int_0^{2\pi} \Psi_c(\Phi + g_n) \sin(p\theta + r_p) d\theta \right),$$

$$(41) \quad \dot{r}_p = \frac{p}{p + am} \left(-\frac{1}{2}p + \frac{a}{\pi A_p} \int_0^{2\pi} \Psi_c(\Phi + g_n) \cos(p\theta + r_p) d\theta \right)$$

for $p = 1, \dots, n$ and (10) and (11).

One can prove the following result.

THEOREM 5.1. *The controller of the form (36) that uses first n modes of the stall stabilizes the system of $2n + 2$ ODEs consisting of the Galerkin projection of (9) onto first n -modes of g and (10) and (11).*

Proof. We are going to use a backstepping controller design, almost identical to the one used for the full model.

Step 1. As a control Lyapunov function for (40) and (41) one uses

$$V_1^n(\Phi, \Psi, g_n) := \frac{1}{2} \|g_n\|_{H_K^1}^2 = \frac{1}{2} \sum_{p=1}^n \left(1 + \frac{am}{p} \right) (pA_p)^2.$$

One has

$$\begin{aligned} \frac{d}{d\xi} V_1^n(\Phi, \Psi, g_n) &= \sum_{p=1}^n \left(1 + \frac{am}{p} \right) \left(p^2 A_p \frac{d}{d\xi} A_p \right), \\ &\frac{a}{\pi} \int_0^{2\pi} \left(\Psi_c(\Phi + g_n) \sum_{p=1}^n p^2 A_p \sin(p\theta + r_p) d\theta \right). \end{aligned}$$

Integrating by parts, one gets

$$\begin{aligned} \frac{d}{d\xi} V_1^n(\Phi, \Psi, g_n) &= \frac{a}{\pi} \int_0^{2\pi} \left(\Psi_c'(\Phi + g_n) \frac{dg_n}{d\theta} \sum_{p=1}^n p A_p \cos(p\theta + r_p) d\theta \right) \\ &= \frac{a}{\pi} \int_0^{2\pi} \Psi_c'(\Phi + g_n) \left(\frac{dg_n}{d\theta} \right)^2 d\theta. \end{aligned}$$

Steps 2 and 3 of the backstepping procedure are exactly the same as in the case of the full model (with g replaced with g_n). \square

Simulations. In this section we illustrate the action of a truncated H^1 -controller with some simulations. The full Moore–Greitzer model has been simulated using 64 Fourier modes (128 states) to represent the stall cell dynamics. The compressor characteristics are assumed to be a cubic function [15]

$$\Psi_c(\phi) := \psi_0 + H \left(1 + \frac{3}{2} \left(\frac{\phi}{W} - 1 \right) - \frac{1}{2} \left(\frac{\phi}{W} - 1 \right)^3 \right).$$

The coefficients ψ_0 , H , and W represent, respectively, the shut-off pressure rise, semi-height, and semiwidth of the characteristic. The parameters ψ_0 , H , W , a , m , l_c , and B determine the compressor model. The Greitzer B parameter determines if the compressor is likely to stall or surge. Stalling compressors are characterized by a low value of the B parameter, while surging compressors are characterized by a high value of the B parameter. We simulated a low B compressor and a high B compressor. We initialized both models with the initial condition for the surge dynamics near the peak of the compressor characteristic. The initial shape of the stall cell was a pure first mode.

Figures 2 and 3 show that, as expected, the state of the uncontrolled low B compressor settled at a rotating stall condition with a significant pressure drop.

The simulations of uncontrolled dynamics are followed by simulations of the dynamics controlled with a truncated H^1 -controller using first four Fourier modes of the stall variable and constant gains. The control function was saturated at 0 to avoid using negative values of the throttle coefficient. We see the state of the low B compressor after a transient period of growing the stall variable and a drop in pressure settled at the desired axisymmetric equilibria near the peak of the compressor characteristic. Figures 4 and 5 show the state evolution.

Figures 6 and 7 show the evolution of the state of the uncontrolled high B compressor. The stall cell initially grows fast, but after the mean mass flow reaches the reverse flow part of the characteristic, it decays. The state of the system undergoes a deep surge cycle.

Figures 8 and 9 show the evolution of the state of the controlled high B compressor. The controller opens the throttle and prevents a transition into a deep surge cycle. Note that the mean flow spends more time in the interval between the well and the peak than in the uncontrolled case. This causes the stall variable to grow and stay large. The stall variable starts to decay only after the mean mass flow becomes bigger than the value corresponding to the peak of the compressor characteristic. This explains why the stall variable decays faster in the uncontrolled case than in the controlled one.

Conclusion. We have constructed a feedback controller stabilizing a peak or any axisymmetric equilibrium to the right of the peak of the compressor characteristic for the full Moore–Greitzer model. The control law resembles control laws for a one-mode truncation of the full model. In the case when the set-point parameter in the controller is such that there is no stable axisymmetric equilibrium we can still guarantee that the dynamics of the closed-loop system are confined to a ball, whose radius can be made arbitrarily small by choosing sufficiently high gains in the controller.

A practical implementation of the H_K^1 -controller would use a finite sum $(\sum_{p=1}^n (1 + \frac{am}{p})(pA_p)^2)^{\frac{1}{2}}$. We proved that this truncated feedback controller actually globally stabilizes the system of $2n+2$ ODEs consisting of the Galerkin projection of the PDE describing the stall onto its first n -modes and two ODEs describing the surge dynamics.

Simulations: stall and surge dynamics
 Low B , uncontrolled.

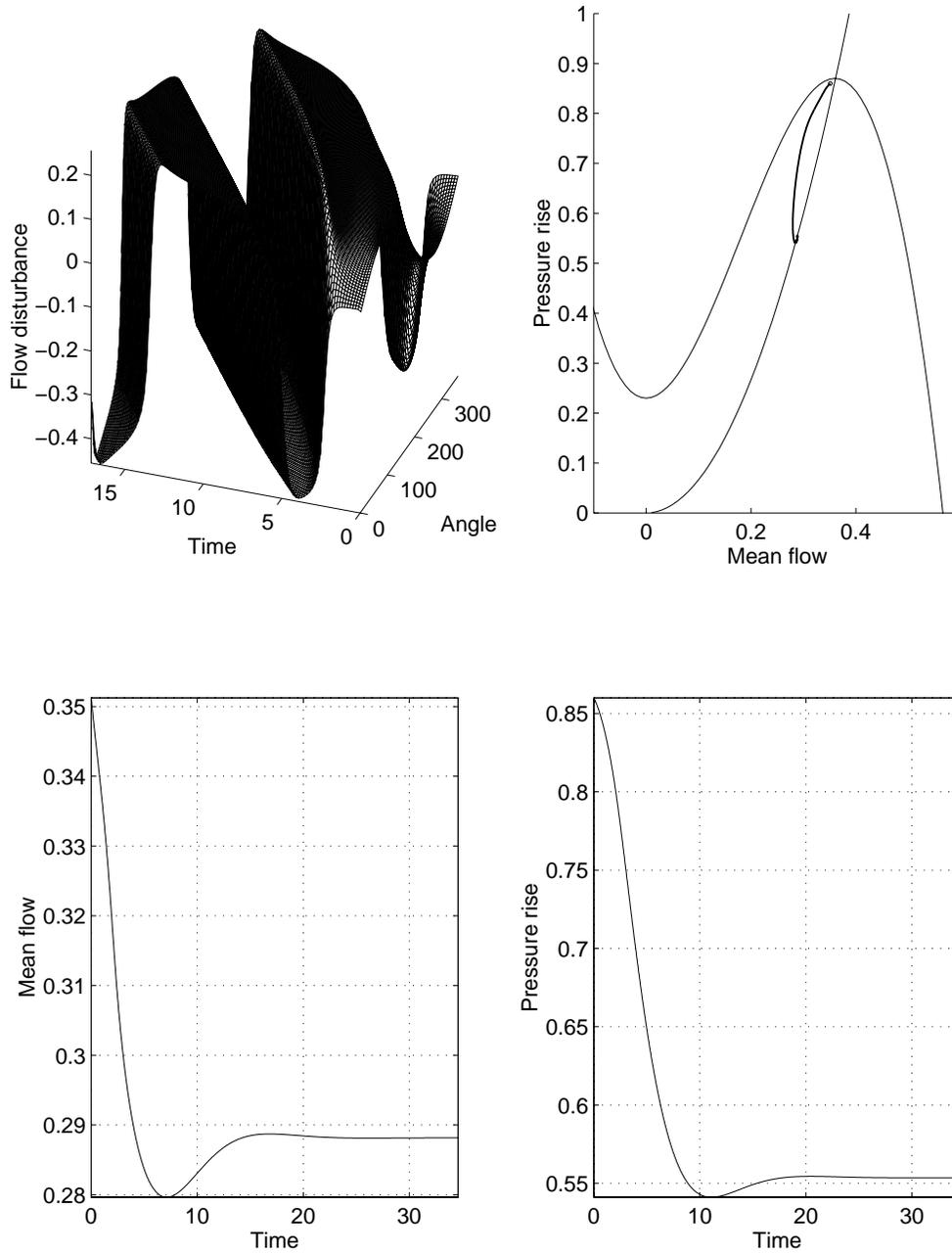


FIG. 2.

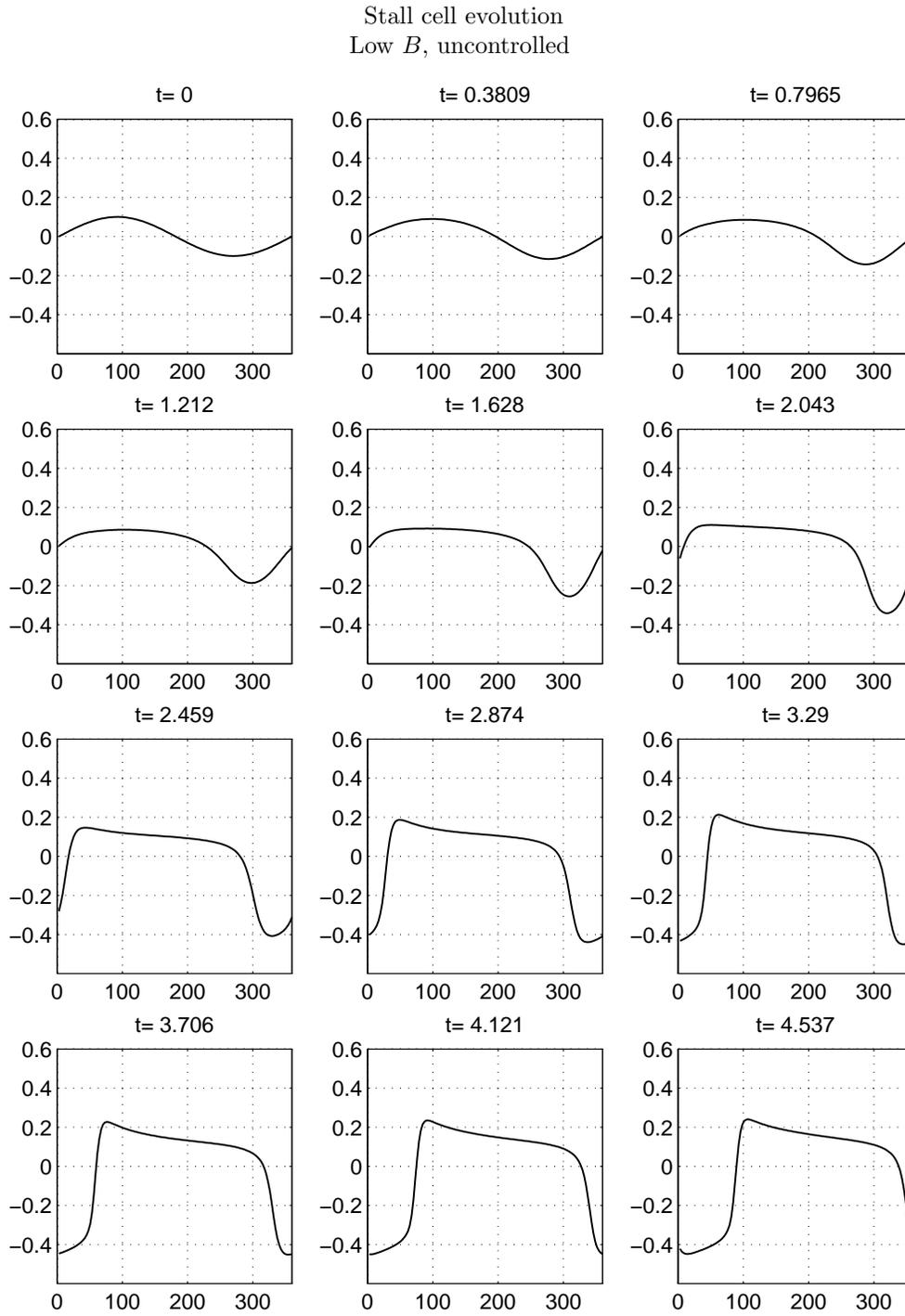


FIG. 3.

Simulations: stall and surge dynamics
 Low B , controlled

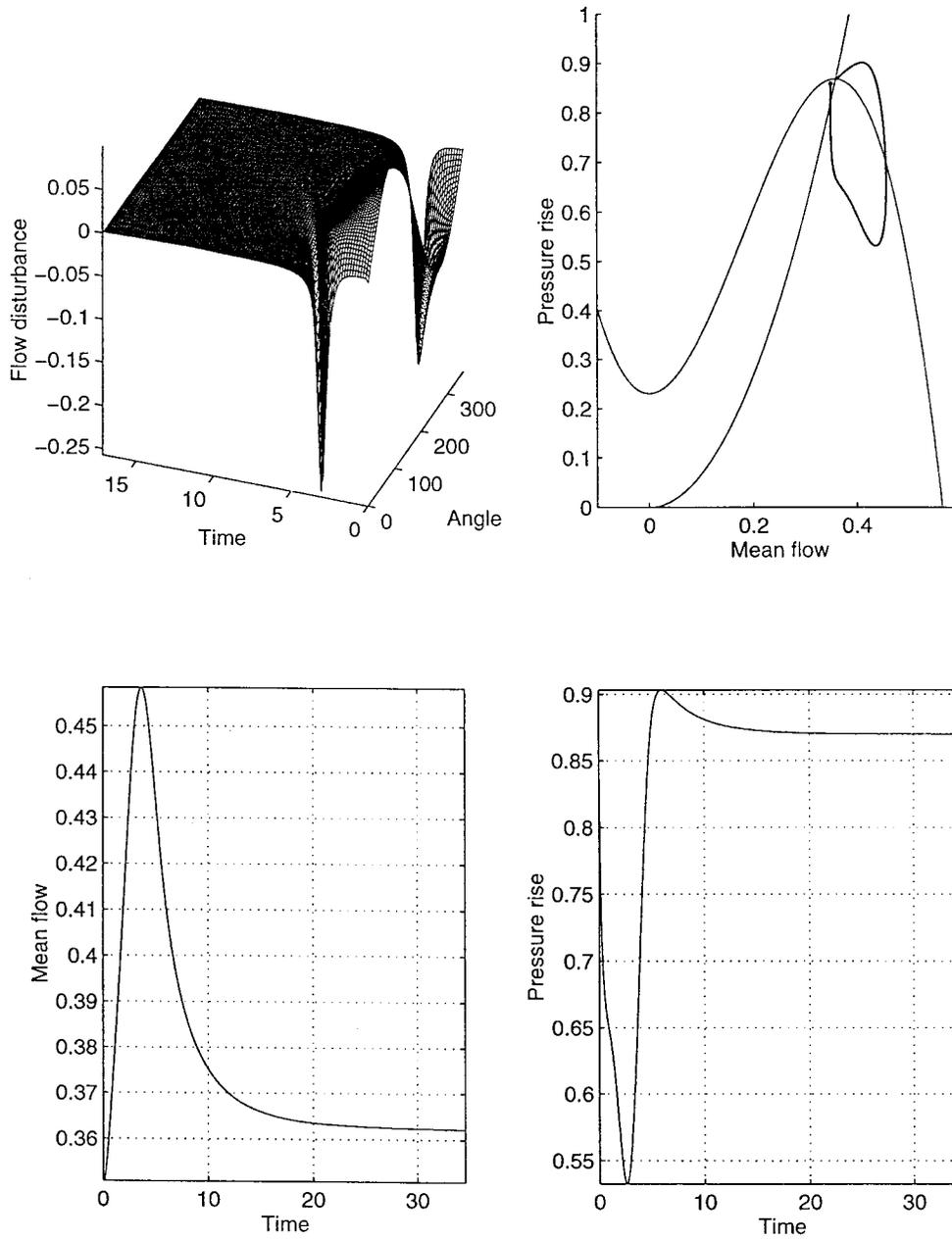


FIG. 4.

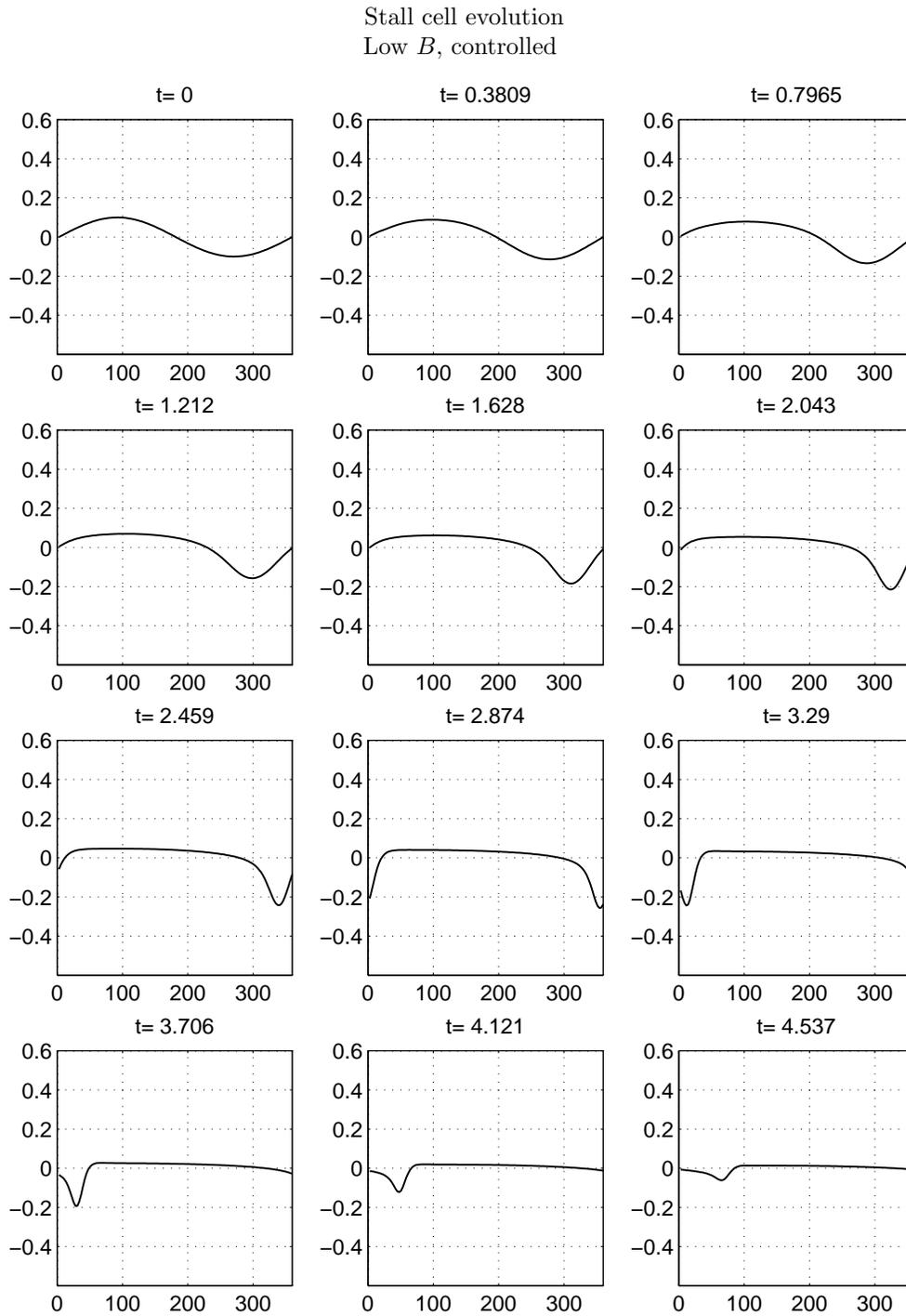


FIG. 5.

Simulations: stall and surge dynamics
High B , uncontrolled

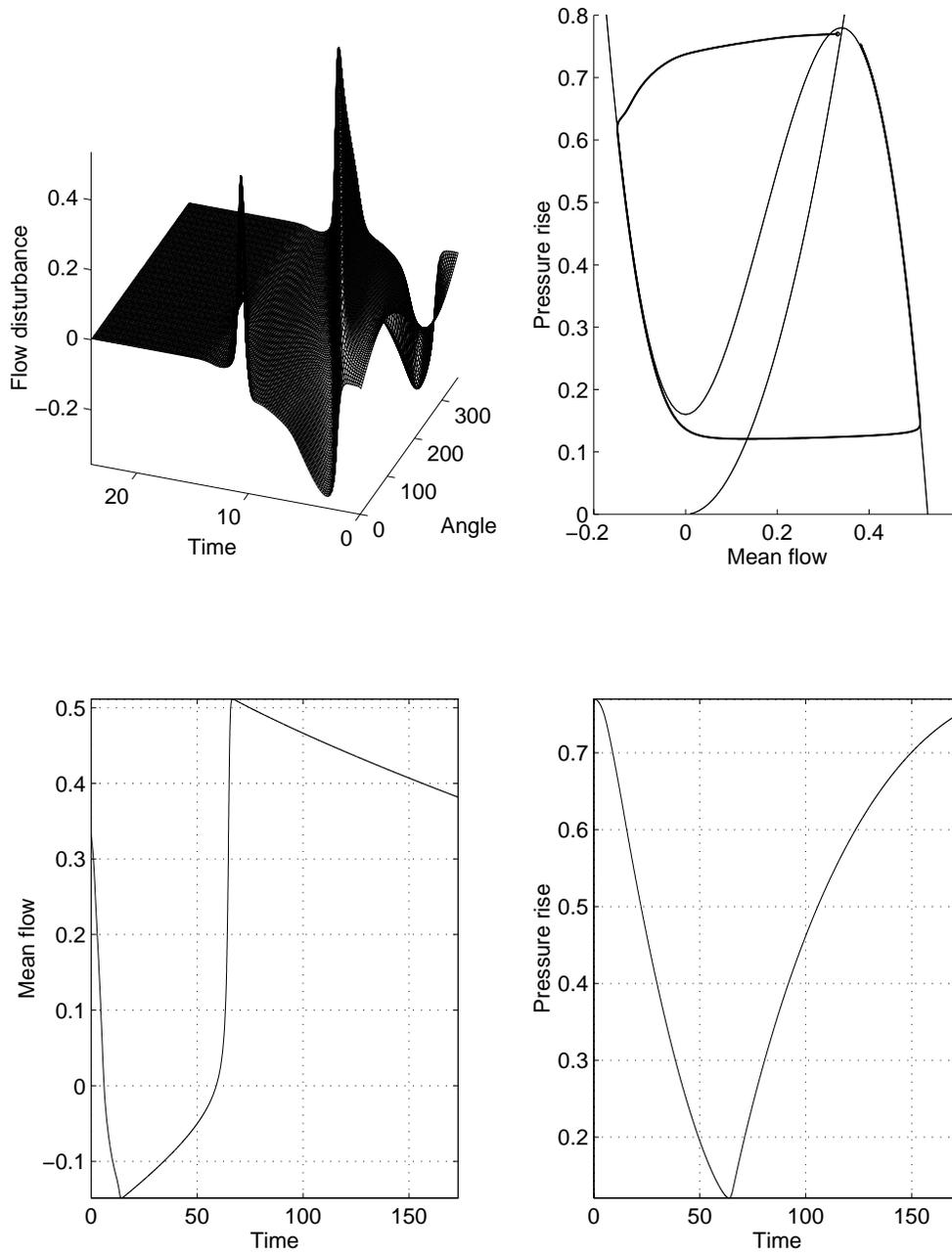


FIG. 6.

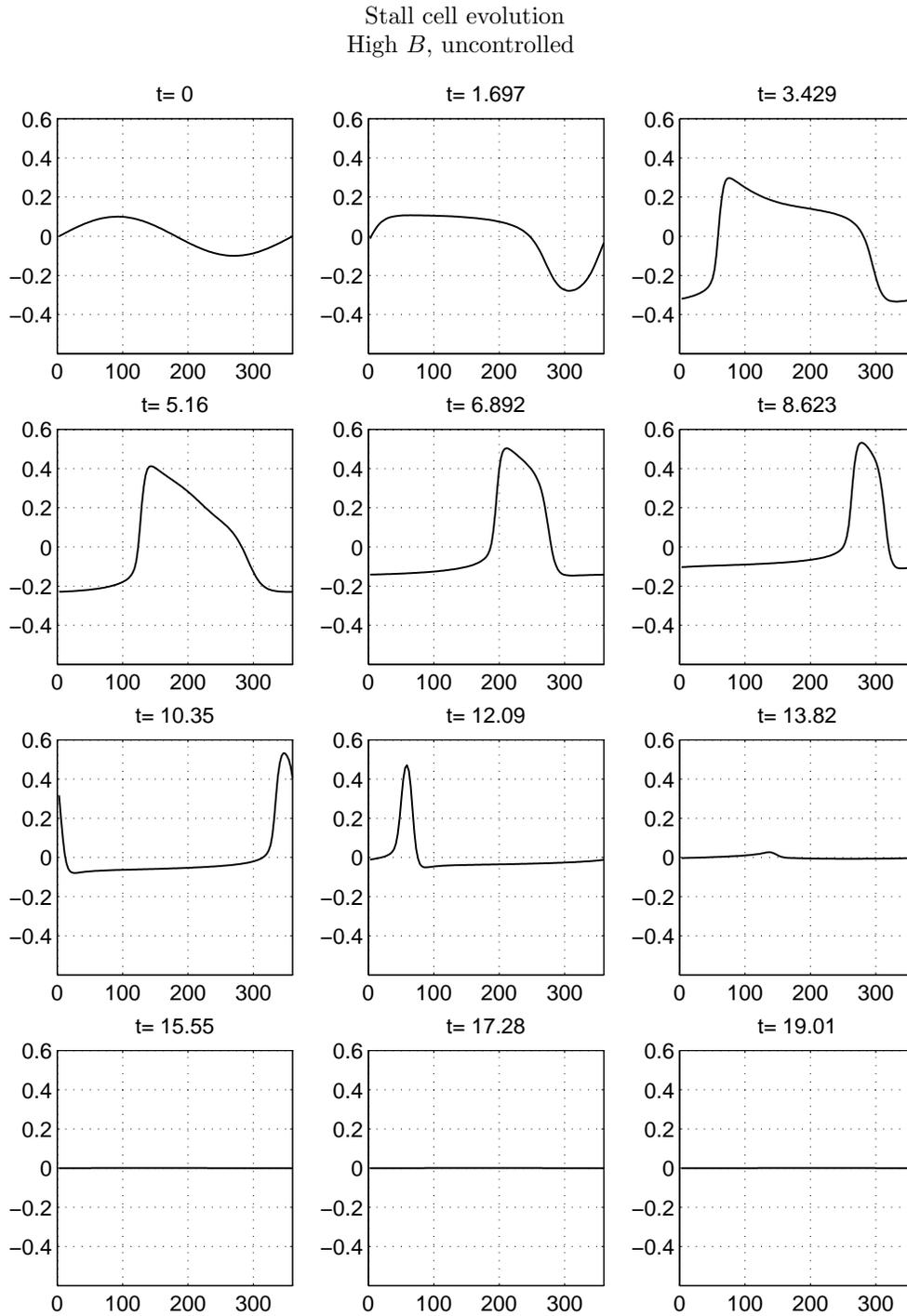


FIG. 7.

Simulations: stall and surge dynamics
High B , controlled

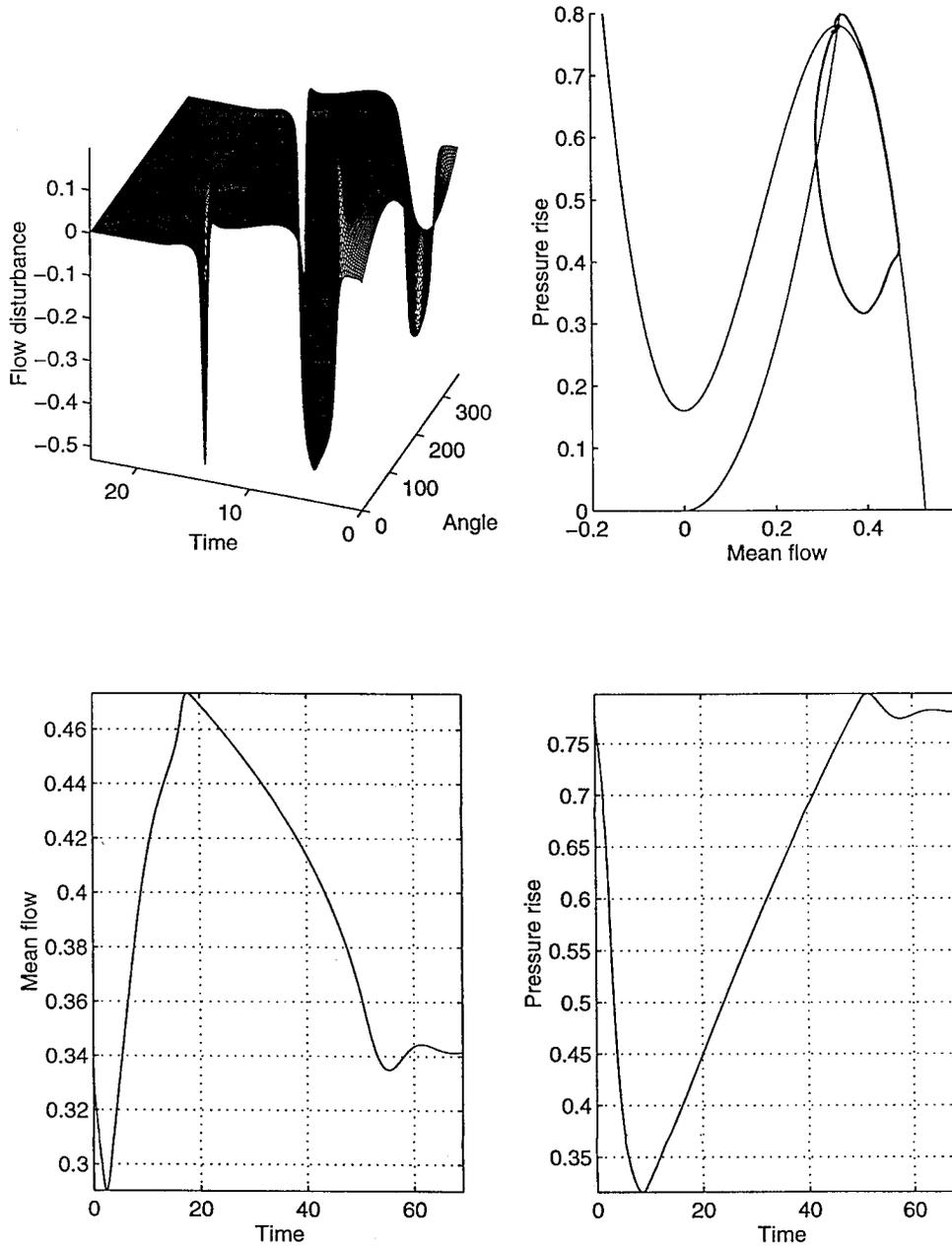


FIG. 8.

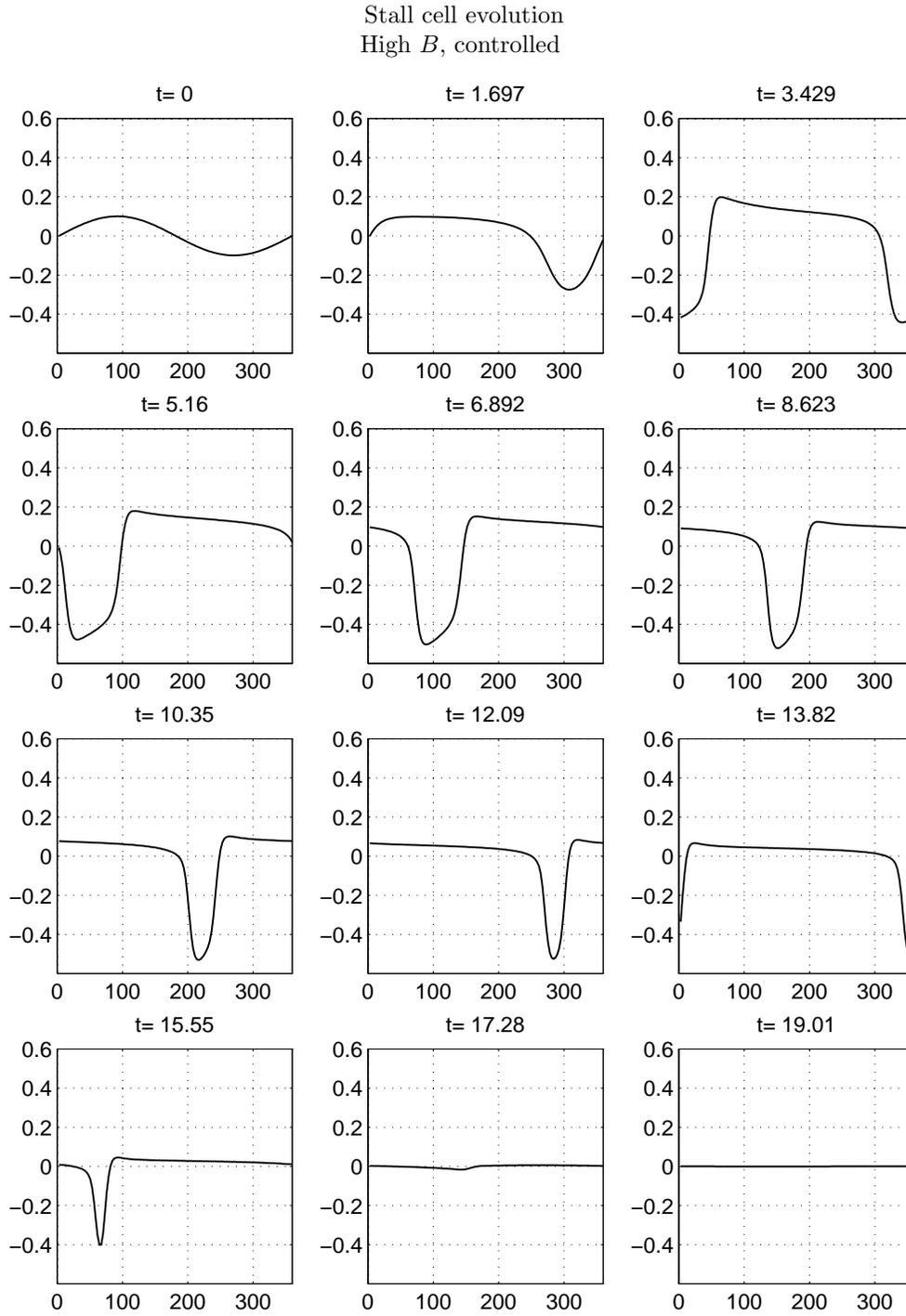


FIG. 9.

While one may argue that the necessity of finding the magnitudes of the Fourier modes of a stall cell for a feedback requires complicated implementation, let us observe that such information would be necessary anyway for any feedback law based on a Galerkin approximation of a Moore–Greitzer PDE model with a finite number of modes.

One feature of the H^1 -controller is not desirable. Namely, its gain increases for higher order modes of the stall cell. This does not seem to be necessary (see Remark 4.1). We conjecture that one can replace the H_K^1 norm of g with the L^∞ norm of g or with the minimum of g in the controller and have the same stabilizability property without using a higher gain for higher order modes of the stall cell. We are currently working on the proof of this conjecture.

Although we have concentrated on a specific model here, the methods developed in this paper can be used, with slight variations, in a variety of problems involving evolution equations.

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