Computation of Koopman spectrum for complex flows

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Data-driven and rigorous analysis of fluid flows?

high-dimensional state space

Koopman viewpoint is to lift the dynamics to the observable space.

\[ f(x) \rightarrow U^t f \circ S^t(x) \]

\[ S^t x \rightarrow S^t(x) \]
Koopman viewpoint is to lift the dynamics to the observable space.

Koopman Operator
\[ U^t \phi = e^{\lambda t} \phi \]

\[ S^t \]
\[ \cdot x \]
\[ \cdot S^t(x) \]
Koopman decomposition can linearize nonlinear dynamical systems.

\[ f(x_0, t) = \sum_{k=1}^{\infty} v_k \phi_k(x_0) e^{\lambda_k t} + \cdots \]

DMD approximates a linear map.

\[
X = \begin{bmatrix}
t_1 & t_2 & \cdots & t_n \\
x_1 & x_2 & \cdots & x_n \\
\end{bmatrix}
\quad Y = \begin{bmatrix}
t_2 & t_3 & \cdots & t_{n+1} \\
x_2 & x_3 & \cdots & x_{n+1} \\
\end{bmatrix}
\]

Compute the linear map \( A \):

\[
Y = AX
\]

and its spectrum,

\[
Av = \lambda v
\]

\( \lambda \): dynamic eigenvalues \( \sim \) Koopman eigenvalues

\( v \): dynamic modes related to Koopman eigenfunctions

DMD uses data vector projections.

- Tu et al., “On DMD, theory and application”, 2013
Dynamics on attractor is usually ergodic.

Measurements on a trajectory:

\[ \tilde{f}(z_0) = [f(z_0), f \circ T(z_0), \ldots, f \circ T^{m-1}(z_0)] \]

\[ \tilde{g}(z_0) = [g(z_0), g \circ T(z_0), \ldots, g \circ T^{m-1}(z_0)]. \]

Then

\[ \lim_{m \to \infty} \frac{1}{m} < \tilde{f}_m(z_0), \tilde{g}_m(z_0) > = \int_A f g^* d\mu, \text{ for a.e. } z_0. \]

- Giannakis, “Data-driven spectral decomposition and forecasting of ergodic dynamical systems”, 2015
To approximate the Koopman operator, we need to represent functions numerically.

Krylov sequence:

$$Uf, U^{t_1}f, \ldots, U^{t_n}f$$

Embedding via Hankel matrix:

$$H = \begin{bmatrix} f(t_0) & f(t_1) & \ldots & f(t_n) \\ f(t_1) & f(t_2) & \ldots & f(t_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ f(t_m) & f(t_{m+1}) & \ldots & f(t_{m+n}) \end{bmatrix}$$

- Brunton et al., “Chaos as an Intermittently Forced Linear System”, 2016
DMD + Ergodicity + Hankel = approximation of Koopman Theorem

Apply DMD to $H$ and $UH$.

Assumptions: Ergodicity, $f$ is in a $k$-dimensional invariant subspace.

Then as $m \to \infty$:

a) $DMD$ eigenvalues $\to$ Koopman eigenvalues.

b) $DMD$ modes $\to$ sampling of Koopman eigenfunctions.

Example: cavity flow at $Re = 16k$

quasi-periodic

snapshot of vorticity

basic Koopman frequencies:

$\omega_1 = 0.9762$, $\omega_2 = 0.6089$

(from FFT+Harmonic average)

kinetic energy

Hankel matrix of kinetic energy:

$H_E = \begin{pmatrix}
E_1 & E_2 & \cdots & E_n \\
E_2 & E_3 & \cdots & E_{n+1} \\
E_3 & E_4 & \cdots & E_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
E_m & E_{m+1} & \cdots & E_{m+n}
\end{pmatrix}$.

Apply DMD to $H_E$ and $UH_E$. 
Hankel+DMD is accurate and fast.

$O(10^{-5})$ error in frequencies.

$O(10^{-2})$ $L^2$—error in eigenfunctions.
Hankel+DMD can be extended to multiple observables.

two observables

Apply DMD to Hankel-block matrices:

\[ H = [H_E, H_v], \quad UH = [UH_E, UH_v] \]
We can compute the asymptotic phase using Hankel+DMD.

What will be the phase difference of $z_1$ and $z_2$?

Find eigenfunction $\phi_0$ associated with $\omega_0$.

Let $\theta = \angle \phi_0$.

**Hankel+DMD finds $\omega_0$ and $\phi_0$ in the same computation!**

Hankel-DMD gives fast, scalable and accurate computation of Koopman spectra from time series.

References:


3) “Data-driven methods for identifying nonlinear models of fluid flows.”, C. Rowley, kitp.ucsb.edu