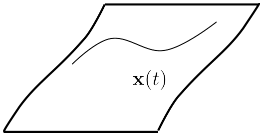
Computation of Koopman spectrum for complex flows

Hassan Arbabi, Igor Mezić
UCSB

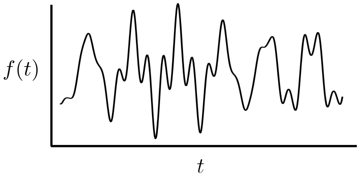
May 2017
SIAM meeting on applied dynamical systems
Snowbird, Utah

Data-driven and rigorous analysis of fluid flows?

high-dimensional state space

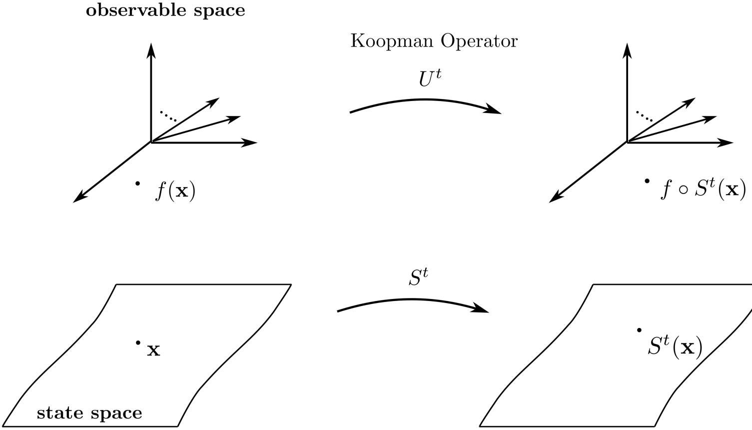


time series

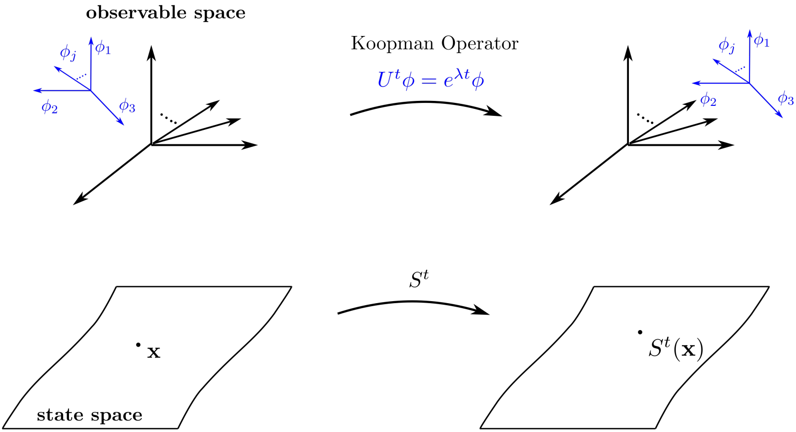


Answer: Koopman Operator Theory,
Dynamic Mode Decomposition (DMD) Algorithm,
Ergodicity Assumption.

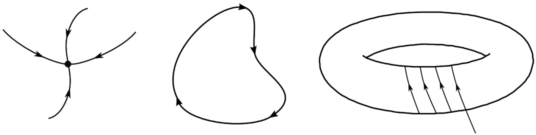
Koopman viewpoint is to lift the dynamics to the observable space.



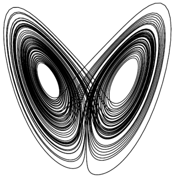
Koopman viewpoint is to lift the dynamics to the observable space.



Koopman decomposition can linearize nonlinear dynamical systems.



$$\mathbf{f}(\mathbf{x}_0, t) = \sum_{k=1}^{\infty} \mathbf{v}_k \phi_k(\mathbf{x}_0) e^{\lambda_k t}$$



$+ \cdots$

- Mezić, "Spectral properties of dynamical systems, model reduction and decompositions", 2005

DMD approximates a linear map.

$$X = \begin{bmatrix} \overset{t_1}{|} & \overset{t_2}{|} & & \overset{t_n}{|} \\ x_1 & x_2 & \ldots & x_n \\ | & | & & | \end{bmatrix} \qquad Y = \begin{bmatrix} \overset{t_2}{|} & \overset{t_3}{|} & & \overset{t_{n+1}}{|} \\ x_2 & x_3 & \ldots & x_{n+1} \\ | & | & & | \end{bmatrix}$$

Compute the linear map A:

$$Y = AX$$

and its spectrum,

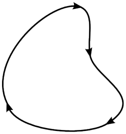
$$Av = \lambda v$$

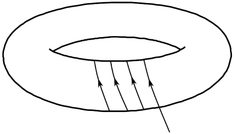
λ: dynamic eigenvalues $\sim$ Koopman eigenvalues
v: dynamic modes related to Koopman eigenfunctions

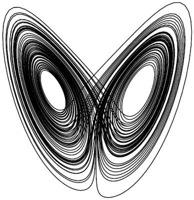
DMD uses data vector projections.

- Tu *et al.*, "On DMD, theory and application", 2013

Dynamics on attractor is usually ergodic.







Measurements on a trajectory:

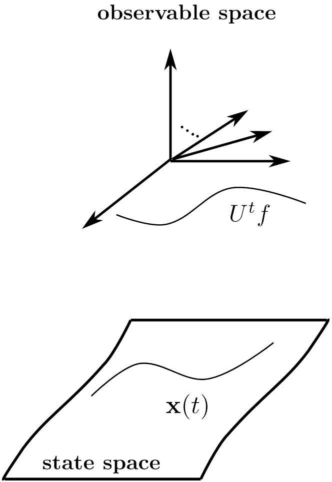
$$\tilde{f}(z_0) = [f(z_0),\ f \circ T(z_0),\ \ \ldots,\ f \circ T^{m-1}(z_0)]$$
$$\tilde{g}(z_0) = [g(z_0),\ g \circ T(z_0),\ \ \ldots,\ g \circ T^{m-1}(z_0)].$$

Then

$$\lim_{m\to\infty} \frac{1}{m} < \tilde{f}_m(z_0), \tilde{g}_m(z_0) >= \int_A fg^* d\mu,\ \text{for a.e. } z_0.$$

- Giannakis, "Data-driven spectral decomposition and forecasting of ergodic dynamical systems", 2015

To approximate the Koopman operator, we need to represent functions numerically.



Krylov sequence:

$$Uf, U^{t_1} f, \ldots, U^{t_n} f$$

Embedding via Hankel matrix:

$$H = \begin{bmatrix} f(t_0) & f(t_1) & \ldots & f(t_n) \\ f(t_1) & f(t_2) & \ldots & f(t_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ f(t_m) & f(t_{m+1}) & \ldots & f(t_{m+n}) \end{bmatrix}$$

- Brunton *et al.*, "Chaos as an Intermittently Forced Linear System", 2016

DMD + Ergodicity + Hankel = approximation of Koopman

Theorem

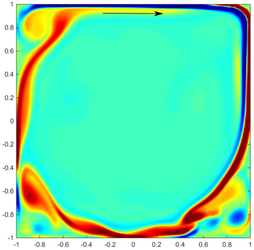
Apply DMD to H and UH.
Assumptions: Ergodicity, f is in a k-dimensional invariant subspace.
Then as $m \to \infty$:

a) *DMD eigenvalues $\to$ Koopman eigenvalues.*
b) *DMD modes $\to$ sampling of Koopman eigenfunctions.*

- Arbabi & Mezić , "Ergodic Theory, DMD and Computation of Koopman Spectral Properties", 2016

Example: cavity flow at $Re = 16k$

quasi-periodic



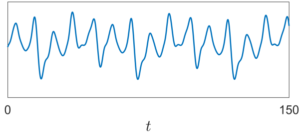
snapshot of vorticity

basic Koopman frequencies:

$\omega_1 = 0.9762, \; \omega_2 = 0.6089$

(from FFT+Harmonic average)

kinetic energy



Hankel matrix of kinetic energy:

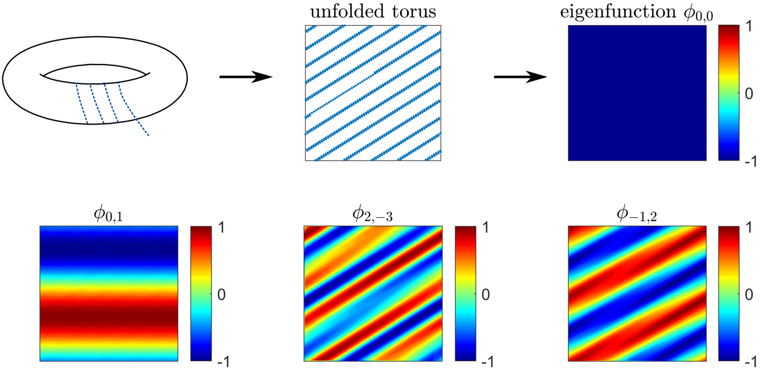
$$H_E = \begin{pmatrix} E_1 & E_2 & \dots & E_n \\ E_2 & E_3 & \dots & E_{n+1} \\ E_3 & E_4 & \dots & E_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ E_m & E_{m+1} & \dots & E_{m+n} \end{pmatrix}.$$

Apply DMD to H_E and UH_E.

Hankel+DMD is accurate and fast.

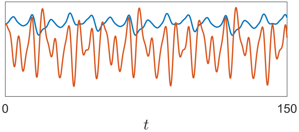
$\mathcal{O}(10^{-5})$ error in frequencies.

$\mathcal{O}(10^{-2})$ L^2-error in eigenfunctions.



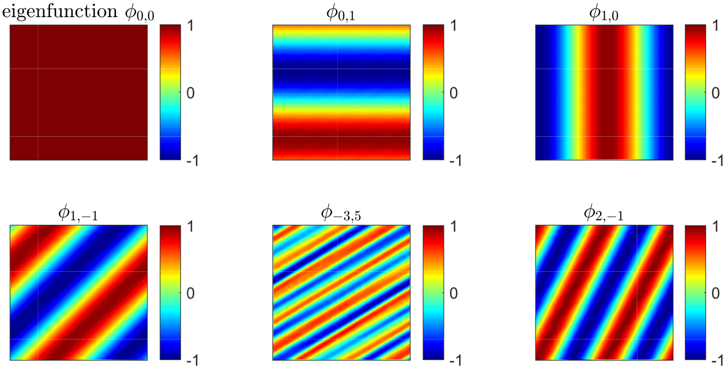
Hankel+DMD can be extended to multiple observables.

two observables

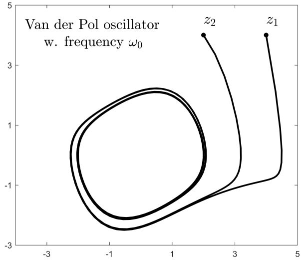


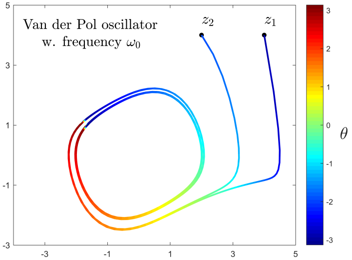
Apply DMD to Hankel-block matrices:

$$H = [H_E, H_v], \quad UH = [UH_E, UH_v]$$



We can compute the asymptotic phase using Hankel+DMD.





What will be the phase difference of z_1 and z_2?

Find eigenfunction ϕ_0 associated with ω_0.

Let $\theta = \angle\phi_0$.

Hankel+DMD finds ω_0 and ϕ_0 in the same computation!

- Mauroy & Mezić , "On the use of Fourier average to compute global isochrons of quasi-periodic attractors", 2012

Hankel-DMD gives fast, scalable and accurate computation of Koopman spectra from time series.

References:

1) "Ergodic Theory, Dynamic Mode Decomposition and computation of spectral properties of the Koopman operator", H. Arbabi and I. Mezić, *arXiv:1611.06664*, 2016.
2) "Study of dynamics in unsteady flows using Koopman Mode Decomposition", H. Arbabi and Igor Mezić, *arXiv:1704.00813*, 2017.
3) "Data-driven methods for identifying nonlinear models of fluid flows.", C. Rowley, **kitp.ucsb.edu**
4) "On Convergence of Extended Dynamic Mode Decomposition to the Koopman Operator", M. Korda and I. Mezić, *arXiv:1703.04680*, 2017.